25 July 2013
Calculus 3, Interphase 2013
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Problem Set 4 [Revised]

Due: **Wednesday 31 July 2013** in class.

1. (15 points) A probability distribution is a nonnegative function that integrates up to 1. The normal distribution is perhaps the most important probability distribution.
   (a) In 1d, $\phi_1(x) = c_1 \exp(-\frac{x^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of c_1 such that $\int_{-\infty}^{\infty} \phi_1(x)dx = 1$.
   (b) In 2d, $\phi_2(x, y) = c_2 \exp(-\frac{x^2+y^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of c_2 such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_2(x, y)dxdy = 1$.
   (c) In 3d, $\phi_3(x, y, z) = c_3 \exp(-\frac{x^2+y^2+z^2}{2\sigma^2})$ is a normal distribution if it integrates up to 1. Find the value of c_3 such that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_3(x, y, z)dxdydz = 1$.

2. (20 points) Moments of inertia for planar objects.
   (a) Consider the triangular shape between $(0, 0, 0), (1, 0, 0), (0, 1, 0)$. Suppose it has constant density ρ. What is its moment of inertia about the y axis?
   (b) What is its moment of inertia about the z axis?
   (c) Consider the disk of radius R, lying in the xy plane. It's center is at $(R, 0, 0)$, and it has constant density ρ. What is it's moment of inertia about the z axis? Use polar coordinates. In polar coordinates, this circle is given by $r(\theta) = 2R\cos\theta$ for $-\pi/2 \le \theta \le \pi/2$.
   (d) What is its moment of inertia about the y axis? Use polar coordinates.

3. (10 points)
   (a) Describe a sphere of radius R in terms of cylindrical coordinates. That is, specify a range of θ, a range of z, and a possibly z- and θ-dependent range of r corresponding to all points in the sphere.
   (b) Describe a cylinder of radius R and height H in terms of spherical coordinates. Locate the cylinder so that its base is on the xy plane. Specify a range of θ, a range of ϕ, and a possibly θ- and ϕ dependent range of r corresponding to all points in the cylinder. Your answer should involve a piecewise function.

4. (15 points) Find the volume and center of mass of the tetrahedron between $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1)$. Assume the tetrahedron has constant density.

5. (10 points) Find the moment of inertia of a constant-density sphere rotating about any axis containing its center. Express the answer in terms as cMR^2, where M is the sphere's mass. Choose a coordinate system wisely.

6. (15 points) *The Curse of Dimensionality*

   (a) Find the average value of the distance to the origin for the 1d points $-R \leq x \leq R$.

   (b) Find the average value of the distance to the origin for the 2d disk of radius R.

   (c) Find the average value of the distance to the origin for the 3d sphere of radius R.

   (d) What do you think is the average distance to the origin for a high dimensional hypersphere of radius R? Does this surprise you?

7. (15 points) *The Olive Problem.* An olive can be formed by taking the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

   and removing the part that is inside the cylinder $x^2 + y^2 = c^2$. Assume $c < a$.

   (a) Make a 3d sketch of the olive and label it with a, b, and c. Make a rz sketch of the olive, as per cylindrical coordinates.

   (b) Compute the volume of the olive.