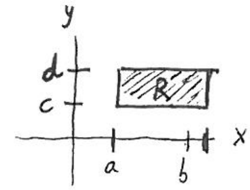
Lecture 8 19 july 2013

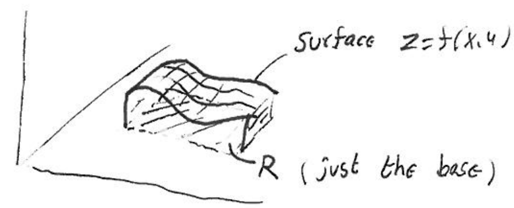
Double Integrals as Volume
Double Integrals as Riemann Sums
Double Integrals as iterated integrals
Double Integrals over complicated regions
Other interpretations of double integrals

Double Integrals and Volume

Let $R = [a,b] \times [c,d]$ be rectangle



Consider surface $z = f(x,y)$.



Volume under surface & over R

is $\iint_R f(x,y)\, dA$

double integral over R

small bit of area

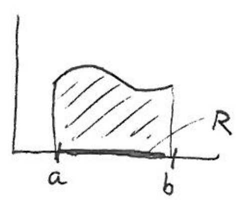
Analogy to 1d:

Consider curve $y = f(x)$

Let $R = [a,b]$

Area under curve & over R

is $\int_R f(x)\, dx$

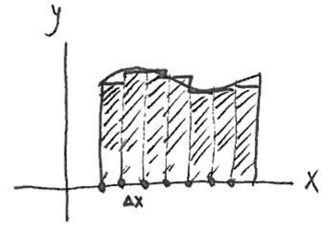


Caution: If f is negative, it counts as negative area/volume.

Riemann Sums

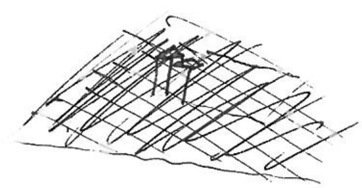
Riemann Sums in 1d

$$\int_{[a,b]} f(x)\,dx \approx \sum f(x_i)\Delta x$$



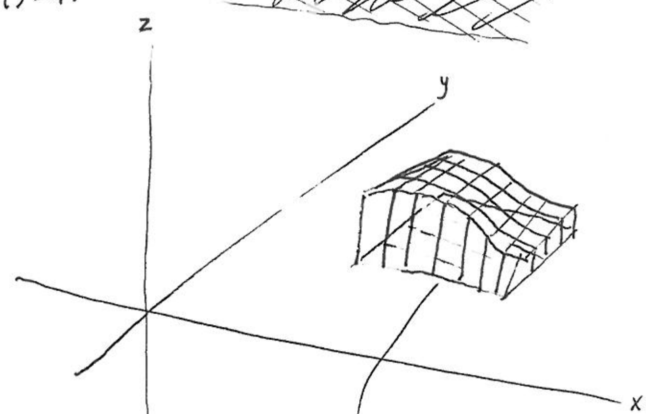
Break region into small strips, add up area of each strip.

Riemann Sums in 2d



- $$\iint_{[a,b]\times[c,d]} f(x,y)\,dA \approx \sum f(x_i,y_i)\Delta A$$

- As $\Delta A \to 0$, get better approximation of overall volume



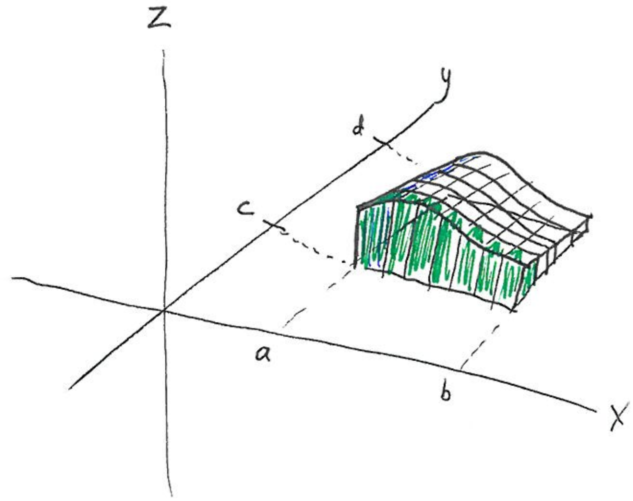
Each column has height $f(x_i,y_i)$ and area ΔA.

Double Integrals as Iterated integrals

$$\iint_{[a,b]\times[c,d]} f(x,y)\, dA = \int_{x=a}^{b} \Bigg(\underbrace{\int_{y=c}^{d} f(x,y)\, dy}_{\substack{x \text{ is constant.} \\ \text{This is 1d integral whose value depends on } x}} \Bigg) dx = \int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\, dx \right) dy$$

integrand has no y dependence (it was integrated away), so this is function of x only. 1d integral

Why?



Partition into rectangles of size $\Delta x\, \Delta y$

Select one row of these columns. (green)

Volume of this part $\approx \int_a^b f(x,c)\, dx \cdot \Delta y$

Row corresponding to y has volume $\approx \int_a^b f(x,y)\, dx \;\; \Delta y$

Adding up these over all y's gives $\sum \left(\int_a^b f(x, y_i)\, dx \right) \Delta y \approx \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$

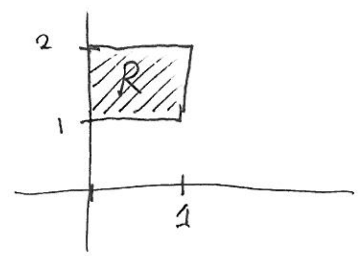
Computing double integrals over rectangles

If $R = [a,b] \times [c,d]$, compute

$$\iint_R f(x,y)\, dA \quad \text{by iterated integral}$$

- $$\int_{x=a}^{b} \left(\int_{y=c}^{d} f(x,y)\, dy \right) dx$$ OR
- $$\int_{y=c}^{d} \left(\int_{x=a}^{b} f(x,y)\, dx \right) dy$$

Example: Find $$\iint_{[0,1]\times[1,2]} xy \, dA$$



- Sketch region of integration
- Write as iterated integral

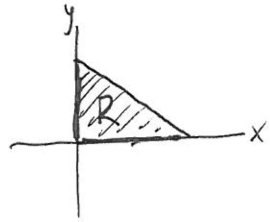
$$\int_{x=0}^{1} \Big(\underbrace{\int_{y=1}^{2} xy \; dy}_{\substack{x \text{ is constant, so} \\ \text{pull it outside}}} \Big) \, dx$$

$$x \int_{y=1}^{2} y \, dy = x \, \tfrac{1}{2} y^2 \Big|_1^2 = x\left(2 - \tfrac{1}{2}\right) = \tfrac{3}{2} x$$

$$= \int_{x=0}^{1} \tfrac{3}{2} x \, dx = \tfrac{3}{2} \, \tfrac{1}{2} x^2 \Big|_0^1 = \boxed{\tfrac{3}{4}}$$

Double Integrals over complicated regions

$$\iint_R f(x,y)\, dA$$



To compute double integral,

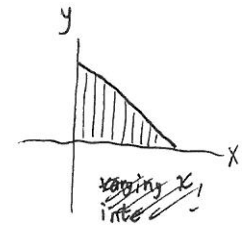
describe region as

- (I) x varies from x_{min} to x_{max} & at fixed x, y varies from $y_{min}(x)$ to $y_{max}(x)$

OR

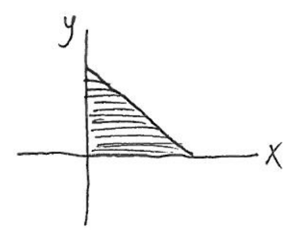
- (II) y varies from y_{min} to y_{max} & at fixed y, x varies from $x_{min}(y)$ to $x_{max}(y)$

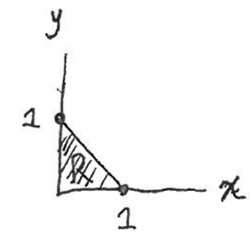
$$\iint_R f(x,y)\, dA = \int_{x_{min}}^{x_{max}} \left(\int_{y_{min}(x)}^{y_{max}(x)} f(x,y)\, dy \right) dx$$



OR

$$\iint_R f(x,y)\, dA = \int_{y_{min}}^{y_{max}} \left(\int_{x_{min}(y)}^{x_{max}(y)} f(x,y)\, dx \right) dy$$



Example: $I = \iint_R xy \, dA$ with 

Describe region:

x ranges from 0 to 1

At fixed x, y ranges from 0 to $1-x$

OR

y ranges from 0 to 1

At fixed y, x ranges from 0 to $1-y$

Write as iterated integral

$$I = \int_{x=0}^{1} \left(\int_{y=0}^{1-x} xy \, dy \right) dx$$

$$= \int_{x=0}^{1} x \left(\int_{y=0}^{1-x} y \, dy \right) dx$$ b/c x is constant in inner integral

$$= \int_0^1 x \, \tfrac{1}{2}(1-x)^2 \, dx$$

$$= \tfrac{1}{2} \int_0^1 (x^3 - 2x^2 + x) \, dx$$

$$= \tfrac{1}{2} \left[\tfrac{1}{4} x^4 \Big|_0^1 - \tfrac{2}{3} x^3 \Big|_0^1 + \tfrac{1}{2} x^2 \Big|_0^1 \right]$$

$$= \frac{1}{24}$$

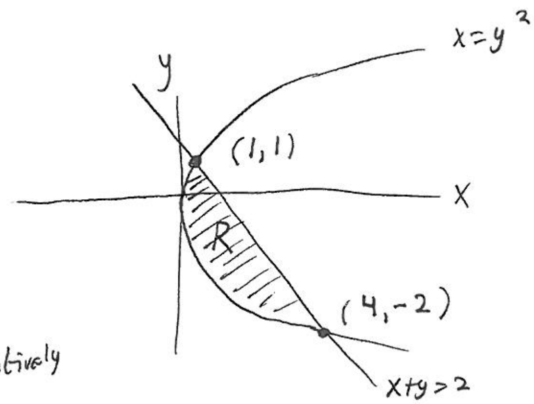
Example:

Set up iterated integral for

$I = \iint_R xy^2 \, dA$ where R is region

between curves $x = y^2$ & $x + y = 2$

(1) Draw region:



(2) Describe region Quantitatively

Complicated to describe as $y_{max}(x)$ & $y_{min}(x)$.

Easy to describe as $x_{max}(y)$ $x_{min}(y)$

y varies from -2 to 1

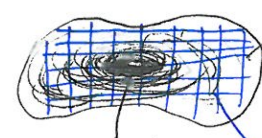
At fixed y, x varies from y^2 to $2-y$

$$I = \int_{y=-2}^{1} \left(\int_{y^2}^{2-y} xy^2 \, dx \right) dy$$

which could be solved.

Other interpretations of double integrals

Let $\rho(x,y)$ be (mass) area-density of an object. (kg/m^2)



$$\text{Mass} \approx \sum \rho(x_i,y_i)\,\Delta x\Delta y$$

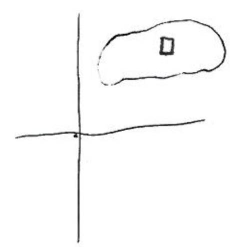
$$= \iint \rho(x,y)\,\underbrace{dx\,dy}_{dA}$$

Let $\rho(x,y)$ be the (number) area-density of bacteria ($\#/m^2$)

$$\text{Total } \# \approx \sum \rho(x_i,y_i)\,\Delta x\Delta y$$

$$= \iint \rho(x,y)\,dA$$

Let $\rho(x,y)$ be (mass) area density.



Moment of inertia about origin

$$I = \sum \rho(x,y)\,\underbrace{(x^2+y^2)}_{r^2}\,dx\,dy$$

m

$$= \iint_R \rho(x,y)\,(x^2+y^2)\,dA$$

Let $\rho(x,y)$ be (mass) area-density

Center of mass: $\vec{\overline{X}} = \dfrac{\iint_R \vec{X}\,\rho(\vec{X})\,dA}{\iint_R \rho(\vec{X})\,dA}$

In terms of components

$$(\bar{x}, \bar{y}) = \left(\frac{\iint_R x\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} , \frac{\iint_R y\,\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA} \right)$$