Lecture 4 - 8 july 2013

Intersections of Lines & Planes

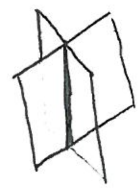
Parametric Form of Lines

Parametric Curves

Velocity, Speed, Acceleration, Tangent Vector, Arclength

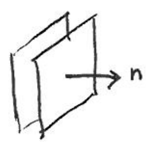
Intersection of Planes & Lines

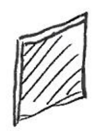
Typically, intersection of two planes is a line



Exceptions: No intersection or complete intersection

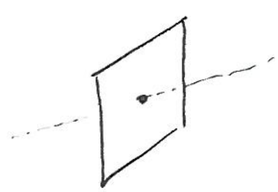
parallel planes — same plane



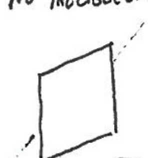


same normal vector

Typically, intersection of plane & line is a point



Exceptions: No intersection or intersection is whole line



- When tangent vector to line perp to normal vector of plane

Typically, intersection of 3 planes is a point.

The typical object of intersection ~~can be~~ is given by counting <u>degrees of freedom</u>.

- The # dimensions of an object (point / line / plane) is # d.o.f. — think: # of ~~numbers~~ coordinates needed to specify a point
- Each (nonredundant) equation reduces d.o.f. by 1.

Examples:

In 3 space, # d.o.f $= 3$

A plane in 3 space: all points satisfy one eqn $\vec{n} \cdot \vec{x} = b$.

3 d.o.f − 1 d.o.f = 2 dof ~ plane

Intersection of two planes: must satisfy 2 eqns.

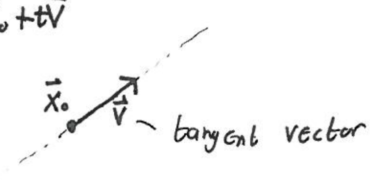
3 dof − 2dof = 1 d.of — line

Intersection of 3 planes: 3 dof − 3 dof = 0 dof — point

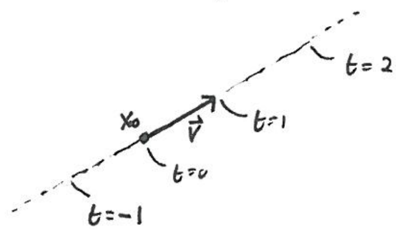
Parametric Form of Lines

- The line (in 2d or 3d) going through $\vec{X}_0$ with tangent vector $\vec{V}$ is given by

$$\vec{X}(t) = \vec{X}_0 + t\vec{V}$$

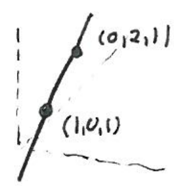


- t is the "parameter". It specifies where along the line you are. t may or may not be the arc length.



- ~~To find~~ Two points specify a line

- Example: Find the line going through $(1,0,1)$ & $(0,2,1)$.



~~Need: $\vec{X}_0$ and $\vec{V}$~~

① Find tangent vector $\vec{V}$ and a point on line, $\vec{X}_0$.

$$\vec{X}_0 = (1,0,1)$$

$$\vec{V} = (0,2,1) - (1,0,1) = (-1,2,0)$$

② Write formula

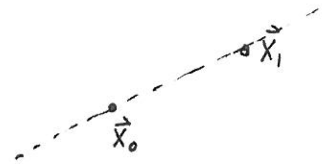
$$\vec{X}(t) = (1,0,1) + t(-1,2,0)$$

- Many parameterizations of a line

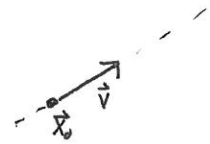
$\vec{X}_0 + t\vec{V}$, $\vec{X}_0 + t(-\vec{V})$, $\vec{X}_0 + t(\vec{V}/2)$, $\vec{X}_1 + t\vec{V}$ for some other point $\vec{X}_1$

Specifying Lines

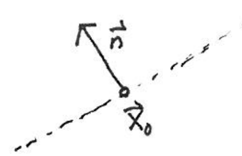
- Line can be specified by two points



- Line can be specified by point + tangent vector



- In 2d, line can be specified by a point
  and a normal vector
  (but not in 3d)



Example: Find the intersection of $\langle 2,3,-1\rangle \cdot \vec{x} = -3$ & $\langle 4,5,1\rangle \cdot \vec{x} = 1$

Planes are not parallel, so intersection is line

To specify a line, need point & tangent vector

① Find point on both planes.

Two eqns & 3 unknowns $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, so set one coordinate arbitrarily.

Let $x_1 = 0$

$$2x_1 + 3x_2 - x_3 = -3$$
$$4x_1 + 5x_2 + x_3 = 1$$

$$\downarrow$$

$$3x_2 - x_3 = -3$$
$$5x_2 + x_3 = 1$$

$$\downarrow$$

$$x_2 = -1/4 \quad \rightsquigarrow \quad \vec{x}_0 = \begin{pmatrix} 0 \\ -1/4 \\ 9/4 \end{pmatrix}$$
$$x_3 = 9/4$$

② Tangent vector.

Any line in plane $\perp$ ~~tangent~~ normal vector of plane.

So line's tangent vector $\perp$ both normal vectors

Cross product gives $\perp$ vector.

$$\vec{v} = \langle 2,3,-1\rangle \times \langle 4,5,1\rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 4 & 5 & 1 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 3 & -1 \\ 5 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$$

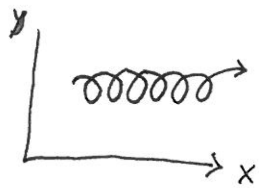
$$= \vec{i}\,(+8) - 6\vec{j} - 2\vec{k} = \langle 8,-6,-2\rangle$$

③ $\vec{x}(t) = \langle 0, -1/4, 9/4\rangle + t\,\langle 8,-6,-2\rangle$

Parametric Curves

How do we specify arbitrary curves?

Eg. Frisbee rotates as it travels. Dot painted on edge.
what shape does it make?

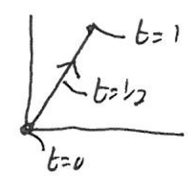


– translation + rotation

Parametric curve is a function $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$ and a range of t
that traces out curve.

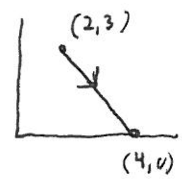
Examples: Line segment from $(0,0)$ to $(2,3)$ given by

$$\vec{x}(t) = t \cdot \langle 2, 3 \rangle \quad \text{for} \quad 0 \le t \le 1$$



Line segment from $(2,3)$ to $(4,0)$ given by

$$\vec{x}(t) = \langle 2, 3 \rangle + t \langle 2, -3 \rangle \quad \text{for} \quad 0 \le t \le 1$$



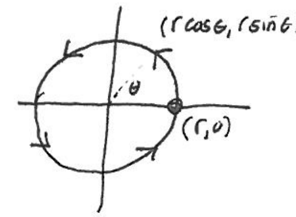
Think of t as time and $\vec{x}(t)$ as position
of object traveling along curve.

The parameter does not need to be time.
Could be $\underbrace{\text{arc length}}_{s}$, $\underbrace{\text{angle}}_{\theta}$, etc.

To find a parameterization, determine where object is
after t time, or s length, or θ angle has occurred.

Example:

Parameterize a circle of radius r counterclockwise with angle as parameter.

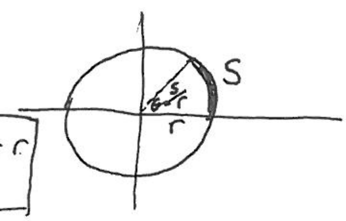


Any point on circle given by $\langle r\cos\theta, r\sin\theta\rangle$

$$\boxed{\vec{X}(\theta) = \langle r\cos\theta, r\sin\theta\rangle \quad \text{for } 0 \le \theta \le 2\pi}$$

Parameterize circle of radius r ... – with arc length as parameter.

Starting at $(r,0)$ if an arc length of S is given, where is point



$\theta = s/r$

$$\boxed{\vec{X}(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right) \quad \text{for } 0 \le s \le 2\pi r}$$

Same but with time. Assume point rotating at angular speed w.

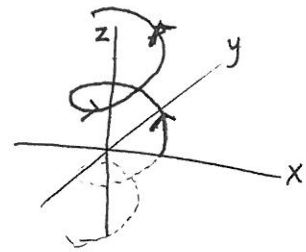
Starting at $(r,0)$, if time t passes, where is point?

$\theta = wt$

$$\boxed{\vec{X}(t) = (r\cos wt, r\sin wt) \quad 0 \le t \le 2\pi/w}$$

Example: helix.

Object rotates in circle w/ angular speed w radius r in x,y plane. Translate w/ speed V in z direction



$$\vec{X}(t) = (r\cos wt, r\sin wt, Vt)$$

for $-\infty < t < \infty$