Agenda 26 July 2013

Triple Integrals in Cartesian

Triple Integrals in Cylindrical Coords

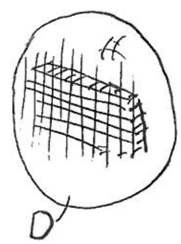
— — — Spherical Coords

Triple Integrals

Let D be a 3d region.

$$\iiint_D f(x,y,z)\, dV$$ is the volume weighted sum of f.

Break into small little cubes $\Delta x \times \Delta y \times \Delta z$



$$\iiint_D f\, dV \approx \sum f(x_i, y_i, z_i)\, \Delta x \Delta y \Delta z$$

Riemann sum

In cartesian, $dV = dx\,dy\,dz$

To evaluate, express region as range of x, a possibly x-dependent range of y, and a possibly x,y-dependent range of z.

(or in any other order of x,y,z)

Interpretations / Applications of Triple Integrals

$\iiint_D 1\, dV =$ Volume of D

$\iiint_D \rho(x,y,z)\, dV =$ mass of D — ρ is spatially dependent density function

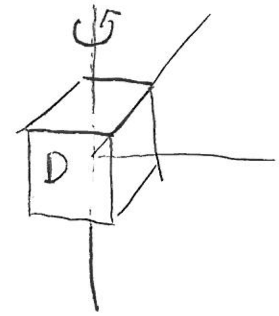
$$\bar{x} = \frac{\iiint_D x\rho\, dV}{\iiint_D \rho\, dV}$$ = x coordinate of center of mass of D (with possibly nonconstant ρ)

$$I = \iiint_D d^2(x,y,z)\, \rho\, dV$$ = moment of inertia about a given axis. $d(x,y,z)$ is distance to axis of rotation

$$\bar{f} = \frac{\iiint_D f(x,y,z)\, dx\,dy\,dz}{\iiint_D dx\,dy\,dz}$$ is average value of f over D.

Example: Find moment of inertia of a cube of width L about the axis shown. Constant density ρ.

Align cube w/ coordinate axes.



$$I = \iiint_D d^2(x,y,z)\, \rho\, dV$$

If rotation axis is z-axis $\quad d(x,y,z) = \sqrt{x^2+y^2}$

$$I = \iiint_D (x^2+y^2)\, \rho\, dx\,dy\,dz$$

Specify D in cartesian coordinates

$$-\tfrac{L}{2} \leq x \leq \tfrac{L}{2}$$
$$-L/2 \leq y \leq L/2$$
$$-L/2 \leq z \leq L/2$$

$$I = \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \rho\,(x^2+y^2)\,dx\,dy\,dz$$

$$= \rho \left[\iiint x^2\,dx\,dy\,dz + \iiint y^2\,dx\,dy\,dz \right]$$

$$= \rho \left[\int_{-L/2}^{L/2} x^2\,dx \int_{-L/2}^{L/2} dy \int_{-L/2}^{L/2} dz + \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} y^2\,dy \int_{-L/2}^{L/2} dz \right]$$

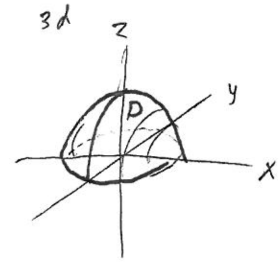
$$= \rho \left[\tfrac{1}{3}x^3 \Big|_{-L/2}^{L/2} L\,L + L\left(\tfrac{1}{3}y^3 \Big|_{-L/2}^{L/2}\right) L \right]$$

$$= \rho \left[\tfrac{2}{3} L^5/8 + \tfrac{2}{3} L^5/8 \right] = \rho \frac{4}{3\cdot 8} L^5 = \frac{1}{6}(\rho L^3) L^2$$

$$= \frac{1}{6} M L^2$$

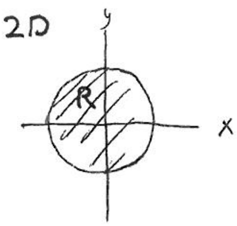
Example: Find volume between $z=0$ plane and $z = 1-x^2-y^2$

3d



$$V = \iiint_D 1 \, dV$$

2D



To evaluate, express V in coordinates.

D is given by (x,y) in R
and $0 \le z \le 1-x^2-y^2$

$$V = \iint_R \left(\int_0^{1-x^2-y^2} dz \right) dx\, dy$$

$$= \iint_R (1-x^2-y^2)\, dx dy$$

How you would have written volume with double integral

Use polar

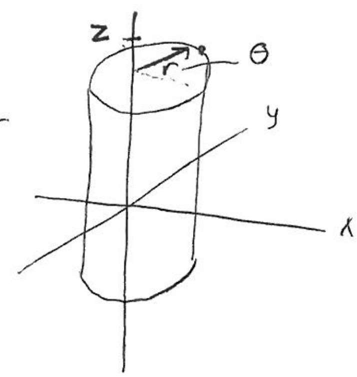
$$V = \int_{\theta=0}^{2\pi}\int_0^1 (1-r^2)\, r dr d\theta = 2\pi \int_0^1 (r-r^3)\, dr = 2\pi \left[1-1/4 \right] = \frac{3\pi}{2},$$

Cylindrical Coordinates

3d

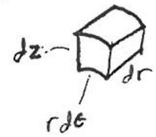
(x, y, z) can be written as (r, θ, z)

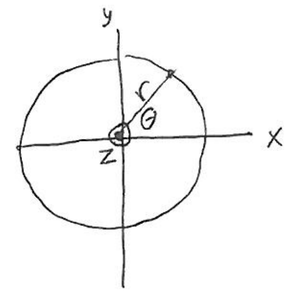
where $(x, y) = (r\cos\theta, r\sin\theta)$ as in polar



Volume element

$dV = r\,dr\,d\theta\,dz$

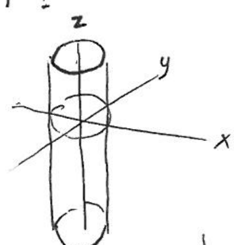




Example:

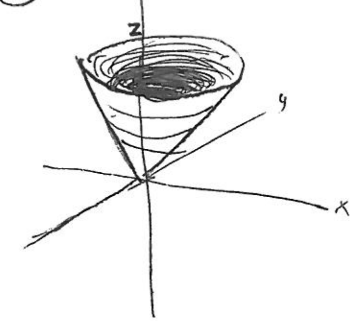
What is shape given by $0 \le r \le 1$?

Infinite cylinder



What is shape given by $z = r$?

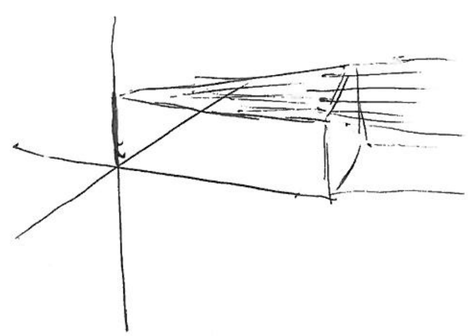
Cone



What is shape given by $0 \le \theta \le \pi/3$

$0 \le z \le 1$

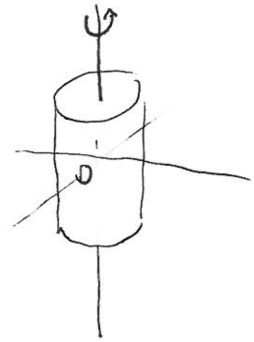
Wedge (infinite) slab



Example:

Moment of inertia of cylinder radius R, length L. about axis of symmetry

Const. density ρ.



$$I = \iiint_D \rho r^2 \, r\,dr\,d\theta\,dz$$

Describe shape

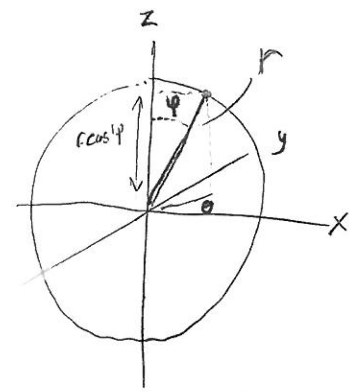
$$0 \le \theta \le 2\pi$$
$$0 \le z \le L$$
$$0 \le r \le R$$

$$I = \int_0^{2\pi} \int_0^{L} \int_0^{R} \rho r^3 \, dr\, d\theta\, dz$$

$$= \rho \int_0^{2\pi} d\theta \int_0^{L} dz \int_0^{R} r^3\, dr$$

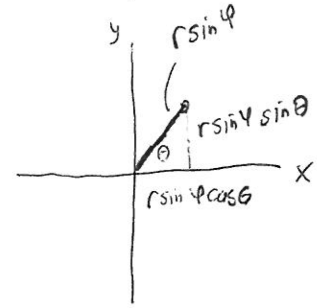
$$= \rho\, 2\pi\, L\, \frac{1}{4} R^4 = \frac{1}{2} (\rho \pi R^2 L)\, R^2 = \frac{1}{2} M R^2$$

Spherical Coordinates



(x, y, z) can be written as (r, θ, φ)

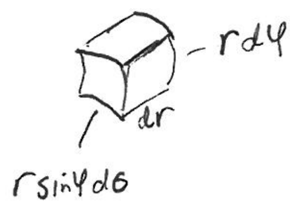
- r - distance to origin
- φ - polar angle (angle from pole) (~latitude
- θ - azimuthal angle (angle about polar axis)



Caution: Physicists use θ for polar angle and φ for azimuthal angle!

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi.$$

Note $x^2+y^2+z^2=r^2$

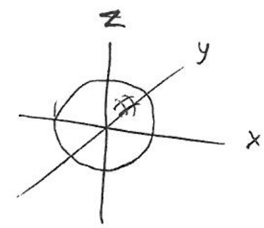


Volume element

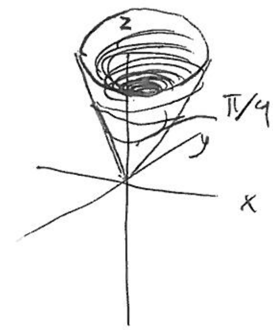
$$dV = r^2 \sin\varphi \; dr\, d\varphi\, d\theta$$

$\uparrow$ polar angle

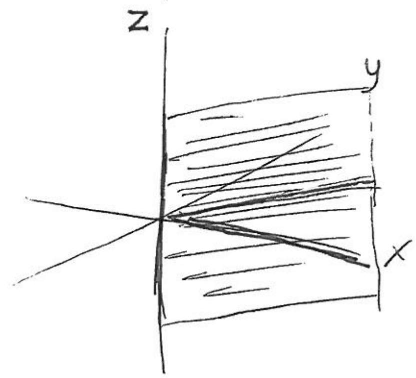
Example $r=1$ describes a sphere



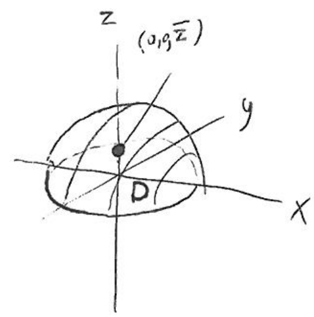
Example: $\varphi=\pi/4$ describes a cone



Example $\theta=\pi/4$ describes a half plane



Example: Find center of mass of hemisphere of radius R.



$$\bar{z} = \frac{\iiint_D z \rho \, dV}{\iiint_D \rho \, dV} = \frac{\iiint_D z \, dV}{\iiint_D dV}$$

Note $\iiint_D dV =$ Volume of $D = \frac{2}{3}\pi R^3$

$$\iiint_D z \, dV = \, ?$$

Describe D in sperical coordinates

$$0 \leq r \leq R$$
$$0 \leq \theta \leq 2\pi$$
$$0 \leq \varphi \leq \pi/2$$

$$\int_{\theta=0}^{2\pi} \int_{\varphi=0}^{\pi/2} \int_{r=0}^{R} r\cos\varphi \; r^2 \sin\varphi \, dr \, d\varphi \, d\theta$$

$$\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi \cos\varphi \, d\varphi \int_0^R r^3 \, dr$$

$$I = 2\pi \; \frac{1}{2}\sin^2\varphi \Big|_0^{\pi/2} \; \frac{1}{4}R^4$$

$$= 2\pi \left(\frac{1}{2}\right) \frac{1}{4} R^4 = \frac{\pi R^4}{4}$$

So $\bar{z} = \frac{3}{8}R$