Day 6 2/2/2015

Activity: Matrices w/ Bernoulli entries

a) $z_i \sim \begin{cases} 1 & \text{w/ } p = 1/2 \\ -1 & \text{w/ } p = 1/2 \end{cases}$

What is σ for $\begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$?

b) How big is $Z \cdot \hat{Z}$ when $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$ $\hat{Z} = \begin{pmatrix} \hat{z}_1 \\ \vdots \\ \hat{z}_m \end{pmatrix}$

and z_i & $\hat{z}_i \sim \begin{cases} 1 & \text{w/ } p = 1/2 \\ -1 & \text{w/ } p = 1/2 \end{cases}$

c) What roughly is Span of $\begin{pmatrix} | & | \\ z & \hat{z} \\ | & | \end{pmatrix}$? for large m

Activity: Matrices w/ Gaussian iid entries

a) What is σ for $\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix}$ where $z_i \sim N(0,1)$ $i = 1 \dots m$

b) Let $z = \begin{pmatrix} z_1 \\ \vdots \\ z_m \end{pmatrix}$ & $\tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \vdots \\ \tilde{z}_m \end{pmatrix}$ be iid $N(0, I_m)$.

How big is $z \cdot \tilde{z}$? is that big or small?

c) What roughly is spectrum $\begin{pmatrix} | & | \\ z & \tilde{z} \\ | & | \end{pmatrix}$ w/ $z, \tilde{z} \sim N(0, I_m)$

Thm: Let A be $N \times n$ w/ iid $N(0,1)$ entries

$\forall\ t \geq 0$,

$$\sqrt{N} - \sqrt{n} - t \leq \sigma_{min}(A) \leq \sigma_{max}(A) \leq \sqrt{N} + \sqrt{n} + t$$

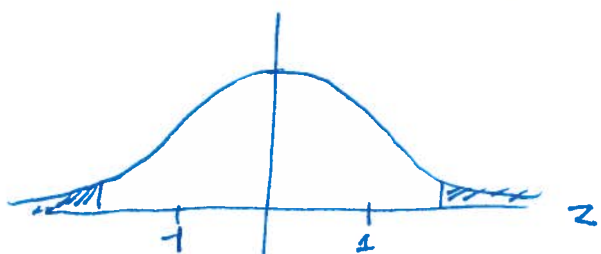
w/ probability at least $1 - 2e^{-t^2/2}$.

"Tall random matrices are approximate isometries"

Note: taking t big provides worse control on singular values but better guarantee on probability bound is true.

Compare to: Let $Z \sim N(0,1)$

$$P(|Z| \in (-t,t)) \geq 1 - \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \quad \text{for } t > 1$$



If we want more certainty on window or Z we need wider window

Theorem: Let $A_{m\times n}$ have iid $N(0,1)$ entries.

Let $\bar{A} = \frac{1}{\sqrt{m}} A$. $\forall k \in [1,n]$ $\forall \delta \in (0,1)$

if $m \geq \frac{C_1}{\delta^2} k \log\left(\frac{en}{k}\right)$ then $\delta_k(\bar{A}) < \delta$

with probability at least $1 - 2e^{-c\delta^2 m}$. Here, C_1, c are universal constants.

Proof: ~~$\forall t \geq 0$~~ ~~$\sqrt{m} - \sqrt{k} - t \leq \sigma_{min}$~~

Consider a fixed subset $T \subset [1,n]$ w/ $|T| = k$

By concentration of singular values,

$$\sqrt{m} - \sqrt{k} - t \leq \sigma_{min}(A_T) \leq \sigma_{max}(A_T) \leq \sqrt{m} + \sqrt{k} + t \quad \text{w/ prob } 1-2e^{-ct^2}$$

So

$$1 - \frac{\sqrt{k}}{\sqrt{m}} - \frac{t}{\sqrt{m}} \leq \sigma_{min}(\bar{A}_T) \leq \sigma_{max}(\bar{A}_T) \leq 1 + \sqrt{\frac{k}{m}} + \frac{t}{\sqrt{m}} \quad \text{w/ prob } 1-2e^{-ct^2}$$

So

$$\|\bar{A}_T^t \bar{A}_T - I_k\| \leq 3\delta_0 \quad \text{where } \delta_0 = \sqrt{\frac{k}{m}} + \frac{t}{\sqrt{m}}$$

where t, m will be st $\delta_0 < \delta/3$

So all subsets of size k are st $\|\bar{A}_T^t \bar{A}_T - I_k\| < \delta$

with probability at least

$$1 - \binom{n}{k} 2e^{-ct^2} \quad \text{provided } \sqrt{\frac{k}{m}} + \frac{t}{\sqrt{m}} < \delta/3$$

$$1 - 2e^{+k\log(en/k) - ct^2} \qquad \text{Note: } \binom{n}{k} \leq \left(\frac{en}{k}\right)^k$$

Choose t st $t = \frac{1}{\sqrt{c}} \sqrt{k \log(en/k)} + \frac{\delta}{6}\sqrt{m}$

So prob. at least

$$1 - 2e^{-c\frac{\delta^2 m}{6}}$$

All we need is $\sqrt{\frac{k}{m}} + C_1' \sqrt{\frac{k \log(en/k)}{m}} < \delta/3$

Choose $m \geq \frac{C_1' k \log en/k}{\delta^2}$

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Application to compressed sensing

To get robust recovery, setting $\delta_{2k}(A) < \sqrt{2} - 1$

If $m \gtrsim \tilde{C}\, k \log n/k$ then A iid $N(0,1)$

satisfies RIP w/ prob $1 - 2e^{-\tilde{c} m}$

If you want a RIP constant twice as good, need 4 times the # measurements. Dont need arb small δ in order to get recovery