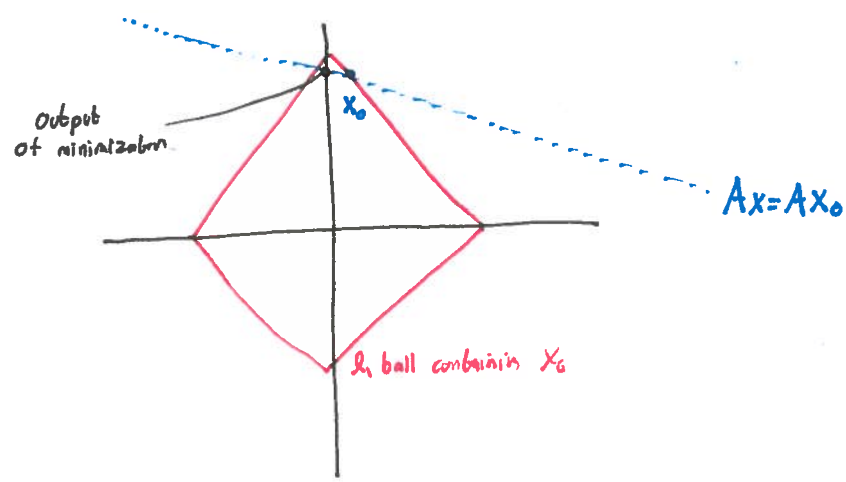
Day 4 1/26/2015

Activity: Consider an $x_0 \in \mathbb{R}^2$ that is close to being sparse. Draw a picture showing that exact recovery by $\min \|x\|_1$ st $Ax = Ax_0$ is not expected, yet approximate recovery is



Theorem: If A is $NSP(2k)$ w/ $C < 1$

then $\hat{x} = \arg\min \|x\|_1$ st $Ax = Ax_0$ satisfies

$\|\hat{x} - x_0\|_2 \leq C_1 \|x_0 - x_{0,k}\|_1$ w/ $x_{0,k}$ is top k coefs of x_0.

Recovery of not-exactly-sparse signals

Claim: If A has NSP(k) w/ $c<1$

then $\hat{x} = \arg\min \|x\|_1$ st $Ax = Ax_0$ satisfies

$$\|\hat{x} - x_0\|_2 \leq C_1 \|x_0 - x_{0,k}\|_1$$ where $x_{0,k}$ is top k coefs of x_0.

Proof: Suppose $\|\hat{x}\|_1 \leq \|x\|_1$. Let $h = \hat{x} - x$.

Let Λ be top k coefs of h.

Then

$$\|h_{\Lambda^c}\|_1 \leq \|h_\Lambda\|_1 + 2\|x_0 - x_{0,k}\|_1 \qquad \text{(optimality)}$$

By NSP

$$\|h_\Lambda\|_2 \leq \frac{c\|h_{\Lambda^c}\|_1}{\sqrt{k}} \leq \frac{c\|h_\Lambda\|_1}{\sqrt{k}} + \frac{2c}{\sqrt{k}}\|x_0 - x_{0,k}\|_1$$

$$\leq c\|h_\Lambda\|_2 + \frac{2c}{\sqrt{k}}\|x_0 - x_{0,k}\|_1$$

So $(1-c)\|h_\Lambda\|_2 \leq \frac{2c}{\sqrt{k}}\|x_0 - x_{0,k}\|_1$

If $c<1$

$$\|h_\Lambda\|_2 \leq \frac{2c}{(1-c)\sqrt{k}}\|x_0 - x_{0,k}\|_1$$

Overall error

$$\|h\|_2 \leq \|h_\Lambda\|_2 + \|h_{\Lambda^c}\|_2$$

Can improve this by $1/\sqrt{k}$ to get $\|h\|_2 \leq C_1 \frac{\|x_0 - x_{0,k}\|_1}{\sqrt{k}}$

$$\leq \|h_\Lambda\|_2 + \|h_{\Lambda^c}\|_1$$

Ok b/c h_{Λ^c} has smaller entries of h.

$$\leq \|h_\Lambda\|_2 + \|h_\Lambda\|_1 + 2\|x_0 - x_{0,k}\|_1 \qquad \text{by optimality}$$

$$\leq (1+\sqrt{k})\|h_\Lambda\|_2 + 2\|x_0 - x_{0,k}\|_1$$

$$\leq \left[\frac{2c}{(1-c)\sqrt{k}}(1+\sqrt{k}) + 2\right]\|x_0 - x_{0,k}\|_1$$

$$\boxed{\|h\|_2 \lesssim C_1 \|x_0 - x_{0,k}\|_1}$$

Finding Sparse structure under noise

Let $x_0 \in \mathbb{R}^n$ $\|x_0\|_0 = k << n.$

Fix $a_i \in \mathbb{R}^n$ $i = 1 \cdots m$

Measure $b_i := a_i \cdot x_0 + e_i$ for unknown x_0 and error e_i

Let $b = Ax_0 + e$ & $\|e\|_2 \leq \varepsilon.$

That is, find sparse x such that $Ax \approx b$

Solve $\min \|x\|_1$ st $\|Ax - b\|_2 \leq \varepsilon.$