

12 January 2015
CAAM 654
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Day 2 — Reading and Questions

Read: Defn 1.1, Thm 1.1, Defn 1.5, Lemma 1.4, Thm 1.7 in Chapter 1 of Eldar and Kutyniok.

1. What is the gist of the argument that control over the spark of a matrix allows a sparse recovery guarantee?
2. If you are handed a matrix, how could you compute its spark? What kinds of matrices have large sparks? Small sparks? The reading says that $\text{spark}(A) \in [2, m + 1]$. Why is that?
3. Formulate a stand-alone precise claim about $m \geq 2k$ being necessary for k -sparse signal recovery. Pay attention to what assumptions you are making on the signal and the measurements. Write the gist of the proof.
4. What sort of measurements have high coherence? Low coherence?
5. Is there an intuitive reason why $\mu(A) \geq 1/\sqrt{m}$ when $n \gg m$?
6. Write a stand-alone precise claim about the minimal number of measurements sufficient for recovering a k sparse signal using a spark and coherence argument. Is this behavior optimal? What would be optimal?

1) Setup: Let $x_0 \in \mathbb{R}^n$, $\|x_0\|_0 = s$.
 Let $A \in \mathbb{R}^{m \times n}$, $m < n$.
 Let $b = Ax_0$.

Find x_0 by solving

$$\min \|x\|_0 \text{ st. } Ax = b. \quad (*)$$

Claim: ^{Fix A.} If $\text{spark}(A) > 2s$, then $\forall x_0$, x_0 is unique soln to $(*)$.

^{Fix A.} If $\text{spark}(A) \leq 2s$, $\exists x_0$ such that x_0 not unique soln to $(*)$

Gist: If x_0 & x_1 are s -sparse solns of $Ax = b$

$$x_0 - x_1 \in N(A) \text{ \& } \|x_0 - x_1\|_0 \leq 2s.$$

Q: Why isn't the following a counter example? ^{to Thm 1.1}

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (**)$$

A has spark ~~2~~ ~~2~~ (take $k=s=1$)
 but there is exactly one soln to $(**)$ with sparsity ≤ 1 :

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A: Spark condition is equivalent to $(*)$ working for all x_0

If spark is too low, $(*)$ may work for some x_0 but will necessarily fail for some other x_0 .

2) How to compute spark exactly?

For $k = 1, 2, \dots$

Try all subsets of size k

If one such subset is dependent, return k .

If $A \in \mathbb{R}^{m \times n}$ & $m < n$, then $\text{spark} \leq m+1$,
b/c any collection of $m+1$ vectors in \mathbb{R}^m are lin'ly dep.

Matrices with large sparks:
Random matrices

How to build matrix w/ spark $m+1$:

Deterministically, add column's 1-at-a-time, ensure lin'ly indep
from all subsets of m existing cols

Randomly

3) Let $m < n$ & $m < 2k$.

Fix $A \in \mathbb{R}^{m \times n}$

$\exists x_0 \in \mathbb{R}^n$ w/ $\|x_0\|_0 \leq k$ such that

x_0 is not a unique minimizer of

$$\min \|x\|_0 \text{ s.t. } Ax = Ax_0. \quad (*)$$

Colloquially: If there are fewer measurements than twice the sparsity, the ℓ_0 -min problem (*) will fail for some signal.

Gist of proof:

If $A \in \mathbb{R}^{m \times n}$, $m < n$, ^{any} ~~there are~~ $m+1$ cols are lin'ly independent.

$\exists y \in \mathcal{N}(A)$ $\|y\|_0 \leq m+1$

write $y = x_0 - x_1$ w/ $\|x_0\|_0 \leq \|x_1\|_0 \approx \frac{m}{2}$

$\Rightarrow Ax_0 = Ax_1$

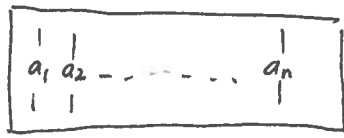
So x_0 is not unique minimizer of

$$\min \|x\|_0 \text{ s.t. } Ax = Ax_0$$

, as x_1 is ~~at least~~ as sparse or ~~more~~ sparse than x_0 .

4)

A is $m \times n$



$a_i \in \mathbb{R}^m$ is column of A .

Low coherence; in \mathbb{R}^2

lowest coherence with 3 vectors in \mathbb{R}^2



~~lowest coherence with 4 vectors~~

lowest coherence with 4 vectors in \mathbb{R}^2 ?

? No coherence = 1 in this case

Large coherence: parallel vectors

5)

A is $m \times n$



$$\mu(A) \geq \sqrt{\frac{n-m}{m(n-1)}} \quad \text{Welch bound}$$

If $n \gg m$, this bound becomes $\mu(A) \geq \frac{1}{\sqrt{m}}$

This bound appears to depend on wrong variable?!!

The ambient dimension

If n is larger, vectors must start "clumping"

So minimal incoherence should be ~~decreasing~~ growing

in n

Resolution: Equality in Welch bound can only occur when $n \leq \binom{m+1}{2}$

~~So the bound $\mu(A) \geq \frac{1}{\sqrt{m}}$ in case $n = m^2$~~

In that case $\mu(A) \geq \frac{1}{\sqrt{m}}$

- ambient dimension controls best incoherence up to a certain n . Then n controls best incoherence

6)

~~For sufficiently large~~

A should be near optimize Welch bound

~~For all m ,~~

$$\exists C \text{ st } \forall k \leq C\sqrt{m} \quad \exists A \in \mathbb{R}^{m \times n} \text{ for } n \leq \frac{m^2}{2}$$

such that $\min \|x\|_0$ st $Ax = Ax_0$ has unique soln x_0 .

Not optimal: want $k \lesssim m$, not $k \lesssim \sqrt{m}$