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CAAM 654
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Day 2 — Reading and Questions

Read: Defn 1.1, Thm 1.1, Defn 1.5, Lemma 1.4, Thm 1.7 in Chapter 1 of Eldar and Kutyniok.

1. What is the gist of the argument that control over the spark of a matrix allows a sparse recovery guarantee?

2. If you are handed a matrix, how could you compute its spark? What kinds of matrices have large sparks? Small sparks? The reading says that spark$(A) \in [2, m+1]$. Why is that?

3. Formulate a stand-alone precise claim about $m \geq 2k$ being necessary for k-sparse signal recovery. Pay attention to what assumptions you are making on the signal and the measurements. Write the gist of the proof.

4. What sort of measurements have high coherence? Low coherence?

5. Is there an intuitive reason why $\mu(A) \geq 1/\sqrt{m}$ when $n \gg m$?

6. Write a stand-alone precise claim about the minimal number of measurements sufficient for recovering a k sparse signal using a spark and coherence argument. Is this behavior optimal? What would be optimal?

1) Setup: Let $x_0 \in \mathbb{R}^n$, $\|x_0\|_0 = s$.
Let $A \in \mathbb{R}^{m\times n}$, $m<n$.
Let $b = Ax_0$.

Find x_0 by solving

$$\min \|x\|_0 \text{ s.t. } Ax = b. \qquad (*)$$

Claim: Fix A. If $\operatorname{spark}(A) > 2s$, then $\forall x_0$, x_0 is unique soln to $(*)$.
Fix A. If $\operatorname{spark}(A) \le 2s$, $\exists\, x_0$ such that x_0 not unique soln to $(*)$

Gist: If x_0 & x_1 are s-sparse solns of $Ax=b$
$x_0 - x_1 \in N(A)$ & $\|x_0 - x_1\|_0 \le 2s$.

Q: Why isn't the following a counter example to Thm 1.1?

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}}_{A}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (**)$$

A has spark 2 (take $k=s=1$)
but there is exactly one soln to $(**)$ with sparsity ≤ 1:
$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A: Spark condition is equivalent to $(*)$ working for all x_0.

If spark is too low, $(*)$ may work for some x_0 but will necessarily fail for some other x_0.

2) How to compute spark exactly?

For $k = 1, 2, \cdots$

Try all subsets of size k

If one such subset is dependent, return k.

If $A \in \mathbb{R}^{m \times n}$ & $m < n$, then spark $\leq m+1$,

b/c any collection of $m+1$ vectors in $\mathbb{R}^m$ are lin'ly dep.

Matrices with large sparks:

random matrices

How to build matrix w/ spark $m+1$:

Deterministically, add column's 1-at-a-time, ensure lin'ly indep from all subsets of m existing cols

Randomly

3) Let $m<n$ & $m<2k$.

Fix $A \in \mathbb{R}^{m \times n}$

$\exists\ x_0 \in \mathbb{R}^n$ w/ $\|x_0\|_0 \leq k$ such that

x_0 is not a unique mizer of

$\min \|x\|_0$ st $Ax = Ax_0$. (*)

Colloquially: If there are fewer measurements then twice the sparsity, the l_0-min problem (*) will fail for some signal.

Gist of proof:

If $A \in \mathbb{R}^{m \times n}$, $m<n$, ~~there are~~ any $m+1$ cols are lin'ly ~~ind~~dependent.

$\exists\ y \in N(A)$ $\|y\|_0 \leq m+1$

write $y = x_0 - x_1$ w/ $\|x_0\|_0, \leq \|x_0\|_0 \approx \frac{m}{2}$

So $Ax_0 = Ax_1$.

So x_0 is not unique minimizer of $\min \|x\|_0$ st $Ax = Ax_0$, as x_1 is ~~at least~~ as sparse or ~~less~~ more sparse.

4) A is $m \times n$ $\quad \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}$

$a_i \in \mathbb{R}^m$ is column of A.

Low coherence:  in $\mathbb{R}^2$

lowest coherence with 3 vectors in $\mathbb{R}^2$

 Coherence is $1/2$

~~lowest coherence with 4 vect~~

lowest coherence with 4 vectors in $\mathbb{R}^2$?

 ? No coherence $= 1$ in this case

Large coherence: parallel vectors

5) A is $m \times n$ $\begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$

$\mu(A) \geq \sqrt{\dfrac{n-m}{m(n-1)}}$ Welch bound

If $n \gg m$, this bound becomes $\mu(A) \geq \dfrac{1}{\sqrt{m}}$

This bound appears to depend on wrong variable ??!!

~~The ambient dimension~~
If n is larger, vectors must start "clumping"
So minimal incoherence should be ~~decaying~~ growing
in n

Resolution: Equality in Welch bound can only occur when $n \leq \binom{m+1}{2}$

~~So bound $\mu(A) \geq$ [illegible] in case $n \geq \frac{m^2}{2}$~~

In that case $\mu(A) \gtrsim \dfrac{1}{\sqrt{m}}$ – ambient dimension controls best incoherence up to a certain n. Then n controls best incoherence

6)

~~For sufficiently lr~~

~~For all m,~~

A shall ~~optimize~~ be near Welch bound

$\exists\, C$ st $\forall\, k \leq C\sqrt{m}$ $\exists\, A \in \mathbb{R}^{m \times n}$ for $n \leq \frac{m^2}{2}$

such that $\min \|x\|_0$ st $Ax = Ax_0$ has unique soln x_0.

Not optimal: want $k \lesssim m$, not $k \lesssim \sqrt{m}$