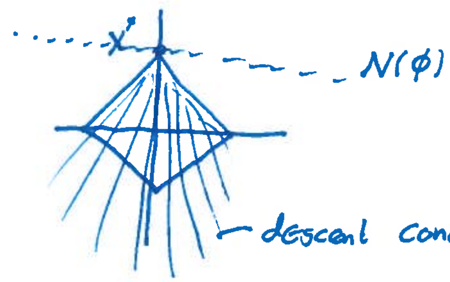
Sparse recovery by Gaussian Widths

Let $x^* \in \mathbb{R}^n$

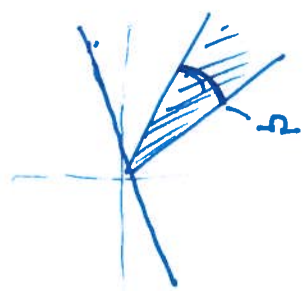
$\min \|x\|_1$ st $\phi x = b = \phi x^*$ ϕ is $m \times n$ with iid $N(0,1)$ entries.

- Recovery succeeds when $Null(\phi)$ misses descent cone of $\|\cdot\|_1$ at x^*.



- Because ϕ gaussian, its null space is $n-m$ dimensional and is uniformly chosen over all $n-m$ dim subspaces

- When does a random subspace miss a ~~cone~~ set Ω?



  If Gaussian width of set $w(\Omega)$ is small, most subspaces will miss it.

- Basis pursuit problem has x^* unique soln whp if

$$m \geq w(\Omega)^2 + 1 \quad w/ \ \Omega = C_{x^*} \cap S^{n-1}$$

$$C_{x^*} = \text{cone}\{h \mid \|x^* + h\|_1 \leq \|x^*\|_1\}$$

$$= \text{descent cone of } \|\cdot\|_1 \text{ at } x^*$$

- What is $w(\Omega)$?

  If $\|x^*\|_0 = s$, $w(C_{x^*} \cap S^{n-1})^2 \leq 2s \log \frac{n}{s} + \frac{5}{4} s$.

- So we need $\approx 2s \log n/s$ measurements.

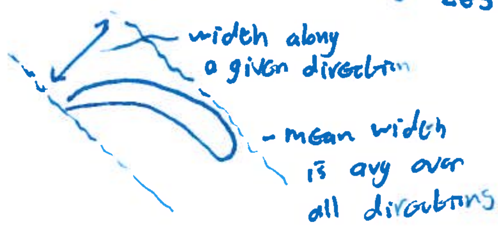
Gaussian Width

Let $S \subset \mathbb{R}^n$. $\quad w(S) = \mathbb{E}_g \sup_{z \in S} \langle g, z \rangle$ w/ $g \sim N(0, I_n)$

How wide is set when projected onto a random ~~direction~~ vector?

If $g \sim \mathrm{Unif}(S^{n-1})$ then $\quad \mathbb{E}_g \sup_{z \in S} \langle g, z \rangle = \frac{1}{2} \int_{S^{n-1}} \max_z \langle g, z \rangle - \min_z \langle g, z \rangle \, dz$

$= \text{mean width}(S)$



$w(S) = \lambda_n \cdot \text{mean width}(S)$ where λ_n is $\mathbb{E}\|Z\|_2$ w/ $Z \sim N(0, I_n)$

Example: $\Omega = S^{n-1}$

$$w(\Omega) = \mathbb{E}_g \sup_{z \in \Omega} \langle g, z \rangle = \mathbb{E}_g \|g\| = \lambda_n \approx \sqrt{n}$$

Example: $\Omega = V \cap S^{n-1}$ $\quad \dim V = k$

$$w(\Omega) = \mathbb{E}_g \|P_V g\|_2 \approx \sqrt{k}$$

Recovery Theorem in terms of Gaussian width

Let ϕ be $m \times n$ with iid $N(0, \frac{1}{m})$ entries

Let $\Omega =$ (descent cone of ℓ_1-norm at x^*) $\cap S^{n-1}$

Thm: x^* is unique soln to $\min \|x\|_1$ st $\phi x = \phi x^*$
with probability at least $1 - e^{-\frac{(\lambda_m - w(\Omega))^2}{2}}$
provided $m \geq w(\Omega)^2 + 1$.

Recall $\lambda_k = \mathbb{E}$(length of k-dim Gaussian $N(0,1)$ entry) & $\frac{k}{\sqrt{k+1}} \leq \lambda_k \leq \sqrt{k}$

Proof:

Background if $f: \mathbb{R}^d \to \mathbb{R}$ lipschitz w/ const L, $\frac{|f(x)-f(y)|}{\|x-y\|} \leq L \quad \forall x, y$

and if $g \sim N(0, I_d)$

$$P[\, f(g) \geq E[f] - t\,] \geq 1 - e^{-t^2/2L^2}$$

For any $\Omega \subset S^{n-1}$ $\quad \Phi \mapsto \min_{z \in \Omega} \|\Phi z\|_2$ is Lip (wrt Frobenius norm)
with $L = 1$

By Gordon's theorem, $\mathbb{E}[\min_{z \in \Omega} \|\phi z\|_2] \geq \lambda_m - w(\Omega)$.

So $\quad P(\min_{z \in \Omega} \|\phi z\|_2 \geq \varepsilon) \geq 1 - e^{-(\lambda_m - w(\Omega) - \sqrt{n}\,\varepsilon)^2/2}$

So $\quad P(\min_{z \in \Omega} \|\phi z\|_2 > 0) \geq 1 - e^{-(\lambda_m - w(\Omega))^2/2}$

Ensure probability is meaningful:

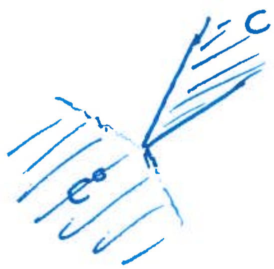
$$\lambda_m \geq \frac{m}{\sqrt{m+1}} \geq \frac{\sqrt{w(\Omega)^2+1}}{\sqrt{1+1/m}} \geq \frac{\sqrt{w(\Omega)^2 + w(\Omega)^2/m}}{\sqrt{1+1/m}} = w(\Omega)$$

Idea of proof:
Concentration of $\min_{z \in \Omega} \|\phi z\|$ away from zero by Gordon's thm
that gives expectation & concentration of Lipschitz functions of Gaussians

Cones & Polar cones

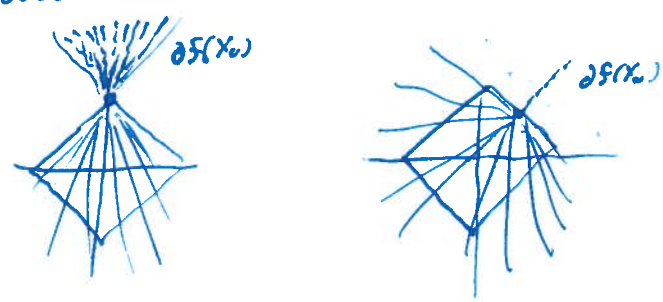
A convex cone is a set C st $x, y \in C \Rightarrow t_1 x + t_2 y \in C \ \forall\, t_{1,2} \geq 0$

The polar cone C° to C is $\{ z \text{ st } z \cdot h \leq 0 \ \forall\, h \in C \}$



If C is cone of descent directions of convex function $f(x)$,
C° given by conic hull of subdifferential of f.

Eg $f(x) = \|x\|_1$



Cor

Polar Cones & Gaussian Width

Let C be convex cone in $\mathbb{R}^n$.

Let $g \sim N(0, I_n)$

$$w(C \cap S^{n-1}) \leq \mathbb{E}_g \, \text{dist}(g, C^\circ)$$

$$\leq \sqrt{\mathbb{E}_g \, \text{dist}^2(g, C^\circ)}$$

$$w^2(C \cap S^{n-1}) \leq \mathbb{E}_g \, \text{dist}^2(g, C^\circ)$$

Proof by convex duality.
(primal problem involves C
dual problem involves C°)

Direct Computation of Gaussian Width of ℓ_1 descent Cone

Fix $x_0^* \in \mathbb{R}^n$ $\|x^*\|_0 = s$.

Let $C = \ell_1$-descent cone at $x^* = \text{cone}(\{h \mid \|x_0^*+h\|_1 \leq \|x^*\|_1\})$

Claim: $w^2(C) \leq 2s \log\frac{n}{s} + \frac{5}{4}s$.

Proof: For simplicity, take x^* such that $\begin{cases} x_i^* > 0 & \text{for } i = 1 \cdots s \\ x_i^* = 0 & \text{for } i > s \end{cases}$

The polar cone C° = subdifferential of $\|\cdot\|_1$ at x_0

$$= \partial\|\cdot\|_1(x_0)$$

$$= \left\{z \;\middle|\; \begin{array}{ll} z_i = t & i = 1 \cdots s \\ |z_i| \leq t & i > s \end{array} \text{ for some } t\right\}$$

By duality, $w^2(C) \leq \mathbb{E}_g \min_{z \in C^\circ} \|g - z\|_2^2$ where $g \sim N(0, I)$

For a fixed g, $\min_{z \in C^\circ} \|g - z\|_2^2 = \inf_t \sum_{i=1}^{s} (g_i - t)^2 + \sum_{i=s+1}^{n} (|g_i| - t)_+^2$

For any t, $\min_{z \in C^\circ} \|g - z\|_2^2 \leq \sum_{i=1}^{s} (g_i - t)^2 + \sum_{i=s+1}^{n} (|g_i| - t)_+^2$

So $\mathbb{E}_g \min_{z \in C^\circ} \|g - z\|_2^2 \leq \mathbb{E} \sum_{i=1}^{s} (g_i - t)^2 + \sum_{i=s+1}^{n} E((|g_i| - t)_+^2)$

$$= s(1 + t^2) + (n - s)\, E(|g_i| - t)_+^2$$

For fixed t, $E(|g_i| - t)_+^2 = \frac{2}{\sqrt{2\pi}} \int_t^\infty (g - t)^2 e^{-g^2/2} dg \leq \frac{2}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$

So $\mathbb{E}_g \min_{z \in C^\circ} \|g - z\|_2^2 \leq s(1 + t^2) + (n - s) \frac{2}{\sqrt{2\pi}} \frac{1}{t} e^{-t^2/2}$

Choose $t = \sqrt{2 \log n/s}$

$$\leq s(1 + 2\log\tfrac{n}{s}) + \frac{s(1 - s/n)}{\pi\sqrt{\log n/s}} \leq 2s \log n/s + \frac{5}{4}s.$$