Activity

Let $X_i \sim N(0,1)$ be indep.
$y_i \sim N(0,1)$

Roughly how big is $\sum_{i=1}^{n} X_i y_i$? How much deviation?

Roughly how big is $\|X\|_2$?

— — — — $\|y\|_2$?

What roughly is angle between X & y?

How much variation is expected.

Angle between random vectors

Let $X \sim N(0, I_n)$
$Y \sim N(0, I_n)$

$$\mathbb{P}(|\langle X, Y\rangle| \geq n\varepsilon) \leq e^{-c\varepsilon^2 n} \quad \text{for } 0 \leq \varepsilon \leq 1$$

Proof: $\langle X, Y\rangle = \sum_i X_i Y_i$

Note $X_i Y_i$ is a subexponential r.v. w/ mean 0.

Apply Bernstein inequality

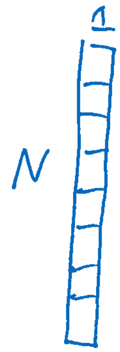
$$\mathbb{P}(|\langle X, Y\rangle| \geq n\varepsilon) \leq 2\exp\left[-c \min\left(\frac{n^2\varepsilon^2}{K^2 n}, \frac{n\varepsilon}{K}\right)\right]$$

$$= 2\exp\left[-c \min\left(\frac{n\varepsilon^2}{K^2}, \frac{n\varepsilon}{K}\right)\right]$$

$$= 2\exp\left[-c\, n\frac{\varepsilon^2}{K^2}\right]$$

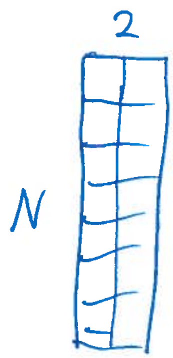
Meaning: Random vectors are nearly orthogonal

Activity



A is $N \times 1$ iid $N(0,1)$ entries

What roughly are sing values of A?



A is $N \times 2$ iid $N(0,1)$ entries

What roughly are sing values of A?

Singular Values of random matrices

Let A be $N \times n$ matrix w/ iid $N(0,1)$ entries.

For any $t \geq 0$, with probability at least $1 - 2e^{-t^2/2}$

$$\sqrt{N} - \sqrt{n} - t \leq \sigma_{min}(A) \leq \sigma_{max}(A) \leq \sqrt{N} + \sqrt{n} + t.$$

(Corollary 5.35 in Vershynin)

Thm 5.32 in Vershynin

Let A be $N \times n$ matrix w/ iid $N(0,1)$ entries. Then

$$\sqrt{N} - \sqrt{n} \leq \mathbb{E}\,\sigma_{min}(A) \leq \mathbb{E}\,\sigma_{max}(A) \leq \sqrt{N} + \sqrt{n}.$$

Proposition 5.34 in Vershynin:

Let $f: \mathbb{R}^n \to \mathbb{R}$ be Lipschitz w/ constant K $\quad |f(x) - f(y)| \leq K\|x-y\|_2 \;\; \forall\, x, y \in \mathbb{R}^n$.

$\forall\, t \geq 0$, $\quad \mathbb{P}(f(X) - \mathbb{E}[f(X)] \geq t) \leq e^{-t^2/2K^2}$

"Lipschitz functions of Gaussian vectors have Gaussian tails

Proof of Corollary 5.35:

$\sigma_{min}(A)$ & $\sigma_{max}(A)$ are 1-Lipschitz functions of A when considered as a vector in $\mathbb{R}^{Nn}$

By Thm 5.32, $\quad \mathbb{E}\,\sigma_{max}(A) \leq \sqrt{N} + \sqrt{n}$.

By Prop 5.34, $\quad \mathbb{P}(\sigma_{max}(A) - \sqrt{n} - \sqrt{n} \geq t) \leq e^{-t^2/2}$

Why is $\sigma_{max}(A)$ 1-Lip? Show $|\sigma_{max}(A+\delta A) - \sigma_{max}(A)| \leq \|\delta A\|_F$.

$$\sigma_{max}(A+\delta A) = \sup_{\substack{u \neq 0 \\ v \neq 0}} \frac{u^t(A+\delta A)v}{\|u\|_2 \|v\|_2} \leq \sup_{\substack{u \neq 0 \\ v \neq 0}} \frac{u^t A v}{\|u\|_2\|v\|_2} + \frac{u^t \delta A v}{\|u\|_2 \|v\|_2} = \sup \frac{u^t A v}{\|u\|\|v\|} + \frac{\langle \delta A, uv^t \rangle}{\|u\|_2 \|v\|_2}$$

$$\leq \sigma_{max}(A) + \|\delta A\|_F \left\| \frac{uv^t}{\|u\|\|v\|} \right\|_F = \sigma_{max}(A) + \|\delta A\|,$$

Thm 5.39 in Vershynin

Let $A \in \mathbb{R}^{N \times n}$ have independent isotropic Sub-Gaussian rows.
Then, $\forall\, t \geq 0$, w/ prob at least $1-2e^{-ct^2}$

$$\sqrt{N} - C\sqrt{n} - t \leq \sigma_{min}(A) \leq \sigma_{max}(A) \leq \sqrt{N} + C\sqrt{n} + t$$

Gist of proof: Covering argument.

Bound $\|Ax\|_2$ for each node ~~over~~ an ε-net

Take union bound

Continuity