Chi Squared Random Variables

A Chi-Squared variable with n d.o.f. is the sum of n squared-Gaussians

If $Z \sim \chi_n^2$ then $Z = \sum_{i=1}^{n} X_i^2$ where $X_i \sim N(0,1)$

Activity: What is typical ~~size~~ value of χ_n^2 (for large n)?

What is typical variation?

If $X \sim \chi_1^2$
What roughly is $P(X > t)$ for large t?

Tail Bounds for Chi Squared Variables

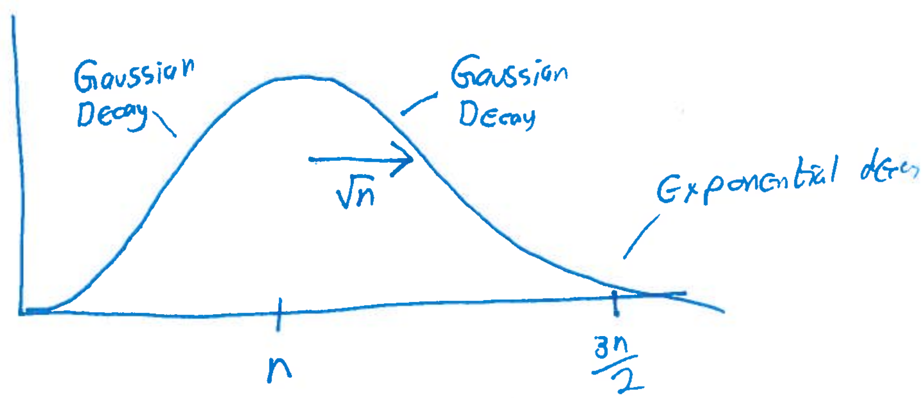
Refs:
Laurent + Massart
Amini + Wainwright

$$\mathbb{P}(\chi_n^2 \leq n - 2\sqrt{n}\,t) \leq e^{-t^2}$$

$$\mathbb{P}(\chi_n^2 \geq n + 2\sqrt{n}\,t + 2t^2) \leq e^{-t^2}$$

$$\mathbb{P}(\chi_n^2 \geq n + \sqrt{n}\,t) \leq e^{-3t^2/16} \quad \text{for } 0 \leq t < \sqrt{n}/2$$

$$\mathbb{P}(\chi_n^2 > n(1+t)) \leq \frac{c}{\sqrt{n}} e^{-nt^2/2} \quad \text{for } 0 \leq t < 1$$



Length of random vectors

Let $X \in \mathbb{R}^n$ be st $X_i \sim N(0,1)$ (aka $X \sim N(0, I_{n \times n})$)

For some c.

$$\mathbb{P}\left(n(1-\varepsilon) \leq \|X\|_2^2 \leq n(1+\varepsilon)\right) \geq 1-2e^{-c\varepsilon^2 n} \quad \text{for } 0<\varepsilon<1/2$$

Proof: Apply $\mathbb{P}(\chi^2_n \geq n + \sqrt{n}\, t) \leq e^{-3t^2/16}$ for $0 \leq t \leq \frac{\sqrt{n}}{2}$

with $t = \sqrt{n}\,\varepsilon$.

Apply $\mathbb{P}(\chi^2_n \leq n - 2\sqrt{n}\, t) \leq e^{-t^2}$ with $t = \frac{\sqrt{n}\,\varepsilon}{2}$.

Meaning: In high dimensions, the length of a random vector is ~~[illegible]~~ ~~very~~ highly certain.

Related: Most of the ~~sphe~~ ball in high dims is located near the surface.

Subexponential Random Variables

A subexponential random variable is one with a tail that is at most exponentially decaying.

A r.v. X is subexponential if for some K_1, K_2

$$\mathbb{P}(|X| > t) \leq e^{1 - t/K_1} \quad \forall t \geq 0$$

OR

$$(\mathbb{E}|X|^p)^{1/p} \leq K_2 p \quad \forall p \geq 1$$

(These are equivalent)

The subexponential norm of X is $\|X\|_{\psi_1} = \sup_{p \geq 1} p^{-1} (\mathbb{E}|X|^p)^{1/p}$

Lemma. Let X be subexponential, $\mathbb{E}[X] = 0$.

For $|t| \leq c/\|X\|_{\psi_1}$ one has

$$\mathbb{E}\, e^{tX} \leq e^{C_1 t^2 \|X\|_{\psi_1}^2}$$

Here, C_1, c are universal constants

Proposition (5.16 in Vershynin) Bernstein ~~type~~ Inequality

Let $X_1 \cdots X_N$ be independent, zero-mean, subexp. rvs and $K = \max_i \|X_i\|_{\psi_1}$. For every $t \geq 0$

$$\mathbb{P}\left(\left| \sum_{i=1}^{N} X_i \right| \geq t \right) \leq 2\exp\left[-c \min\left(\frac{t^2}{K^2 N}, \frac{t}{K} \right) \right]$$

Corollary

$$\mathbb{P}\left(\left| \frac{1}{N} \sum_{i=1}^{N} X_i \right| \geq \varepsilon \right) \leq 2 \exp\left(-c \min\left(\frac{\varepsilon^2}{K^2}, \frac{\varepsilon}{K} \right) N \right)$$

Proof: WLOG, let $K = 1$.

Let $S = \sum_{i} X_i$. Let $\lambda > 0$

$$\mathbb{P}(S \geq t) = \mathbb{P}(e^{\lambda S} \geq e^{\lambda t}) \leq e^{-\lambda t} \mathbb{E}\, e^{\lambda S} = e^{-\lambda t} \prod_i \mathbb{E}[e^{\lambda X_i}]$$

$$\leq e^{-\lambda t} \prod e^{c' \lambda^2} = \exp(-\lambda t + c' \lambda^2 N)$$

Choose $\lambda = \min\left(\frac{t}{2c'N}, c \right)$ we have

$$\mathbb{P}(S \geq t) \leq \exp\left[- \min\left(\frac{t^2}{4c'N}, \frac{ct}{2} \right) \right]$$

Prop. follows by $\mathbb{P}(S \leq -t)$ by symmetry.

Corollary follows by taking $\varepsilon N = t$.