Concentration Estimate. Let $X_i \sim N(0, 1)$ iid.

$$\mathbb{P}\left(\left| \frac{1}{n} \sum_{i=1}^{n} X_i \right| \geq \frac{c}{\sqrt{n}} \right) \leq \frac{2}{c} e^{-c^2/2}.$$

it is very improbable to be many std. dev away from avg.

Hoeffding's Inequality

Let X_i be iid $\mathbb{E}[X_i]=\mu$ & $a \le X_i \le b$. $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \varepsilon\right) \leq 2e^{-2n\varepsilon^2/(b-a)^2}$$

Gaussian decay

$\sigma \simeq \frac{b-a}{2}\frac{1}{\sqrt{n}}$

Activity: Explain what each part of probability bound represents

Pf: Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

WLOG, let $\mu=0$. Let $t>0$.

$$\mathbb{P}(\bar{X} > \varepsilon) = \mathbb{P}\left(\sum_{i=1}^{n} X_i \geq n\varepsilon\right) = \mathbb{P}\left(e^{t\sum_{i=1}^{n} X_i} \geq e^{tn\varepsilon}\right)$$

$$\leq e^{-tn\varepsilon}\,\mathbb{E}\left(e^{t\sum_{i=1}^{n} X_i}\right) \quad \text{by Markov}$$

$$= e^{-tn\varepsilon}\,\mathbb{E}\left(e^{tX_i}\right)^n$$

Technical result: $\mathbb{E}(e^{tX_i}) \leq e^{t^2(b-a)^2/8}$

So $\mathbb{P}(\bar{X} > \varepsilon) \leq e^{-tn\varepsilon} e^{t^2 n(b-a)^2/8}$

choose t to minimize this quantity $\left(t = \frac{4\varepsilon}{(b-a)^2}\right)$

So $\mathbb{P}(\bar{X} > \varepsilon) \leq e^{-2n\varepsilon^2/(b-a)^2}$.

Result follows from $\mathbb{P}(\bar{X} > \varepsilon) + \mathbb{P}(\bar{X} < -\varepsilon)$.