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Analysis I
Paul E. Hand
hand@rice.edu

Week 9 — Summary — Series and Power Series

93. If $\{x_n\}$ is a sequence in a normed vector space, we define the infinite sum $\sum_{n=1}^{\infty} x_n = \lim_{N\to\infty} \sum_{n=1}^{N} x_n$. The infinite series converges if this sum exists. We say that an infinite series diverges if the partial sums are unbounded.

94. Comparison test. Let $\sum a_n$ and $\sum b_n$ be series of real numbers. If $\sum b_n$ converges and $0 \leq a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ converges.

95. Ratio test. Let $\sum a_n$ be a series of nonnegative real numbers, and let $0 < c < 1$ be such that $a_{n+1} \leq c a_n$ for sufficiently large n. Then $\sum a_n$ converges.

96. Integral test. Let f be a decreasing function over all real numbers ≥ 1. The infinite series $\sum_{n=1}^{\infty} f(n)$ converges if and only if $\int_a^{\infty} f(x)dx$ exists and is finite. Note that $\int_a^{\infty} f(x)dx$ is defined as $\lim_{M\to\infty} \int_1^M f(x)dx$.

97. Let $\sum a_n$ be a series of numbers. If $\sum |a_n|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum |a_n|$ converges.

98. Let $\{a_n\}$ be a sequence of numbers monotonically decreasing to zero. The alternating series $\sum (-1)^n a_n$ converges.

99. Let $\sum a_n$ be a series of vectors in a complete normed vector space. If $\sum \|a_n\|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum \|a_n\|$ converges.

100. Let $\sum x_n$ be an absolutely convergent series in a complete normed vector space. Then the series obtained by any rearrangement of the series also converges absolutely to the same limit.

101. We say that an infinite series of functions $\sum_n f_n(x)$ converges absolutely on S if $\sum |f_n(x)|$ converges for all $x \in S$. We say the infinite series converges uniformly on S if the sequence of partial sums converges uniformly on S.

102. Weierstrass test: Let $f_n \in L^{\infty}$ be such that $\|f_n\|_{\infty} \leq M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly and absolutely. If each f_n is continuous, then so is $\sum f_n$.

103. For any power series $\sum a_n x^n$, there is a radius of convergence R (which may be zero, finite, or infinite), such that the series converges absolutely for all $|x| < R$ and does not converge absolutely for any $|x| > R$.

104. The radius of convergence of $\sum a_n x^n$ is $1/\limsup_{n\to\infty} |a_n|^{1/n}$.

105. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence $R > 0$. Then, for all $|x| < R$, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ and this sum converges absolutely for all $|x| < R$.

106. Let $\{f_n\}$ be a sequence of functions in $C^1([a, b])$ and assume that $f_n' \to g$ uniformly, and that $f_n(x_0)$ converges for some x_0. Then, there exists a function f such that $f_n \to f$ uniformly, and f is differentiable, and $f' = g$.

107. Let $f(x) = \sum a_n x^n$ be a power series with radius of convergence $R > 0$. Then, an antiderivative of $f(x)$ in $-R < x < R$ is given by $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ and this sum converges absolutely for all $|x| < R$.