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Analysis I
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Week 2 — Summary — Differentiation, Mean Value Theorem, Taylor Series

23. The derivative of f at x is $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$, if this limit exists. A function is differentiable on a set if it is differentiable at every point in that set.

24. Product rule, quotient rule, chain rule.

25. Differentiability implies continuity.

26. *Let $C^p([a,b])$ be the set of functions defined on $[a,b]$ that are differentiable p times, and the p-th derivative is continuous. Let C^∞ be the set of functions that are in C^p for all p.

27. At a local maximum (or minimum) of a differentiable function, the derivative is zero (provided that this max or min occurs in the interior of the function's domain).

28. *Mean value theorem: If f is continuous on $[a,b]$ and is differentiable on (a,b), then for some $c \in (a,b)$, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

29. Big oh and Little oh notation:

    (a) $f(x) = o(g(x))$ as $x \to x_0$ means that $f(x)/g(x) \to 0$ as $x \to x_0$

    (b) $f(x) = O(g(x))$ as $x \to x_0$ means that there exists C such that $|f(x)| \leq Cg(x)$

30. *A Taylor series is a local approximation of a function, and it is obtained by matching the value and a given number of derivatives of that function at a particular point.

31. *The nth order Taylor series of $f(x)$ about $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$$

32. The nth Taylor remainder term is

$$R_n(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n \right).$$

33. *The nth order Taylor series is accurate to the $n+1$st order in the neighborhood of the point of expansion. The constant factor of the error term is controlled by the maximum value of the $n+1$st derivative of the function.

    If $f \in C^{n+1}$ in a neighborhood of a, then $R_n(x) = O(|x-a|^{n+1})$ as $x \to a$. More precisely,

$$R_n(x) \leq \max |f^{(n+1)}| \cdot \frac{|x-a|^{n+1}}{(n+1)!}.$$

    The max is taken over the neighborhood and the inequality holds for all points in the neighborhood.