27 November 2017
Analysis I
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Coverpage to Pledged HW 11

Time limit: 3 hours. You may not use your books, your homeworks, your notes, or any electronics during the exam. Please write the start and finish times on your paper. To receive full credit, you must name all major theorems and state definitions used in your arguments. All problems must be accompanied by a proof. You may cite results from class and well-known theorems.

This homework is pledged. On the first page, please write your signature and the Rice University pledge: "On my honor, I have neither given nor received any unauthorized aid on this homework."

Due: Thursday, 30 November 2017 in class.

[The exam is on the next page]

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1. Let $C_0^2([0,1])$ be the set of twice-continuously-differentiable functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$. Define

$$\|f\|_a = \sup_x |f(x)| + |f'(x)| + |f''(x)|,$$
$$\|f\|_b = \sup_x |f'(x)| + |f''(x)|,$$
$$\|f\|_c = \sup_x |f(x)| + |f''(x)|.$$

   (a) (10 points) Prove that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms.

   (b) (10 points) Are $\|\cdot\|_a$ and $\|\cdot\|_c$ equivalent norms? Prove your answer.

2. (a) (10 points) Provide a counterexample to the following claim: Let $\mathcal{K}$ be a collection of sets in a normed vector space E such that the intersection of every finite subcollection of sets in $\mathcal{K}$ is nonempty. The intersection of all of the sets in $\mathcal{K}$ is nonempty.

   (b) (10 points) Let $\mathcal{K}$ be a collection of *compact* sets in a normed vector space E such that the intersection of every finite subcollection of sets in $\mathcal{K}$ is nonempty. Prove that the intersection of all of the sets in $\mathcal{K}$ is nonempty.

   Hint: Consider building an open cover of a single set within $\mathcal{K}$.

3. (15 points) Show that there exists an open neighborhood U of $(1,1) \in \mathbb{R}^2$, and $X \in C^1(U, \mathbb{R})$ so that $X(1,1) = 0$ and

$$X(y,z)z + [\sin X(y,z)]y + [\cos X(y,z)]z = 1$$

   for all $(y,z) \in U$.

4. (20 points) Prove or disprove: $\{x \in \ell_1 \mid \sum_{n=1}^{\infty} |x_n| n^2 \leq 1\}$ is a compact subset of ℓ_1 with respect to the ℓ_1 norm.