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Analysis I
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## Week 5 — Summary — Inner Products, Equivalent Norms, Complete Normed Vector Spaces

48. An inner product $\langle \cdot, \cdot \rangle$ satisfies the following axioms for all $u, v, w \in V$:
    (a) $\langle v, w \rangle = \langle w, v \rangle$
    (b) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
    (c) If $c \in \mathbb{R}$, $\langle cv, w \rangle = c\langle v, w \rangle = \langle v, cw \rangle$
    (d) $\langle v, v \rangle \geq 0 \;\forall v$ and $\langle v, v \rangle = 0 \Rightarrow v = 0$.
49. Inner products induce a norm $\|v\| = \sqrt{\langle v, v \rangle}$.
50. *Inner products satisfy the Cauchy-Schwarz inequality $\langle v, w \rangle \leq \|v\|\|w\|$.
51. *Notes from Bill Symes on Dimension Theory (linear independence, basis, dimension). See website.
52. *Definition: Two norms $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent on a vector space $V$ if there exists $c, C > 0$ such that
$$c\|x\|_b \leq \|x\|_a \leq C\|x\|_b \;\forall x \in V.$$
53. *All norms on finite dimensional vectors spaces, e.g. $\mathbb{R}^n$, are equivalent.
54. *In infinite dimensional vector spaces, some pairs of norms are not equivalent.
55. *Definition: A sequence $x_n$ in a normed vector space is Cauchy if
$$\forall \varepsilon \;\exists N \text{ such that } n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon.$$
56. *In a normed vector space, we say that $x_n$ converges to $x$ if $\forall \varepsilon \;\exists N$ such that $n \geq N \Rightarrow \|x_n - x\| < \varepsilon$. We write this as $\lim_{n\to\infty} x_n = x$
57. *Definition: A vector space is complete if any Cauchy sequence converges to an element in the set.
58. *Definition: A Banach space is a complete normed vector space.
59. *Definition: $\mathbb{R}^n$ is a Banach space under the $\ell_\infty$ norm. By equivalence of norms on finite dimensional spaces, it is a Banach space under any norm.