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Analysis I
Paul E. Hand
hand@rice.edu

Incoming ability with proofs

The purpose of these problems is to help me understand your incoming ability with regards to proofs. Please spend no more than an hour at these problems. Please submit your answers to my first floor mailbox by Noon on Monday Aug 24, 2015.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is continuous at x_0 if

$$\forall \varepsilon > 0 \, \exists \, \delta > 0 \text{ such that } |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

1. Show that
$$f(x) = \begin{cases} 1 & x > 0, \\ 0 & x \leq 0. \end{cases}$$
is not continuous at $x = 0$.

2. Prove that the following are equivalent:
$$f \text{ is continuous at } x_0 \Leftrightarrow \forall \varepsilon > 0 \, \exists \, \delta > 0 \text{ such that } |x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon$$

1.) Show that $f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases}$ is not continuous at $x = 0$.

Proof: It suffices to show

$$\exists\, \varepsilon > 0 \text{ s.t. } \forall\, \delta > 0 \quad |x - 0| < \delta \not\Rightarrow |f(x) - 0| < \varepsilon.$$

Let $\varepsilon = \frac{1}{2}$. Fix $\delta > 0$. $x = \delta/2$ is such that $f(x) = 1$.

As $|x - 0| < \delta$ and $|f(x)| \geq \varepsilon$, we conclude $|x - 0| < \delta \not\Rightarrow |f(x) - 0| < \varepsilon$ ∎

2) Prove that the following are equivalent:

(a) $\forall \varepsilon \ \exists \delta_\varepsilon$ such that $|x-x_0|<\delta_\varepsilon \Rightarrow |f(x)-f(x_0)|<\varepsilon$

(b) $\forall \tilde{\varepsilon} \ \exists \tilde{\delta}_{\tilde{\varepsilon}}$ such that $|x-x_0| \leq \tilde{\delta}_{\tilde{\varepsilon}} \Rightarrow |f(x)-f(x_0)| \leq \hat{\varepsilon}$

Proof:

First, we prove $(a) \Rightarrow (b)$.

Fix $\hat{\varepsilon}>0$. Let $\tilde{\delta}_{\tilde{\varepsilon}} = \dfrac{\delta_{\tilde{\varepsilon}}}{2}$.

Observe that $|x-x_0| \leq \tilde{\delta}_{\tilde{\varepsilon}} \Rightarrow |x-x_0|<\delta_{\tilde{\varepsilon}}$

By (a), $|x-x_0|<\delta_{\tilde{\varepsilon}} \Rightarrow |f(x)-f(x_0)|<\tilde{\varepsilon} \leq \hat{\varepsilon}$.

Hence, $|x-x_0| \leq \tilde{\delta}_{\tilde{\varepsilon}} \Rightarrow |f(x)-f(x_0)| \leq \hat{\varepsilon}$.

Second, we prove $(b) \Rightarrow (a)$.

Fix $\varepsilon>0$. Let $\delta_\varepsilon = \tilde{\delta}_{\varepsilon/2}$.

Observe that

$$|x-x_0|<\delta_\varepsilon \Rightarrow |x-x_0| \leq \tilde{\delta}_{\varepsilon/2} \Rightarrow |f(x)-f(x_0)| \leq \frac{\varepsilon}{2} < \varepsilon,$$

where the second implication follows from (b). $\blacksquare$