Given $\epsilon > 0$ select $\delta > 0$ such that if $|x - \alpha| < \delta$ and $p/q = x \in \mathbf{Q}$, then $1/q < c\epsilon$. Then

$$\left|\frac{f(x)}{x-\alpha}\right| \leq \frac{q^2}{c} \cdot \frac{1}{q^3} < \epsilon,$$

thus

$$\lim_{\mathbf{Q} \ni x \to \alpha} \frac{f(x) - f(\alpha)}{x - \alpha} = 0.$$

If x is irrational, then the Newton quotient is 0, so f is differentiable at α and $f'(\alpha) = 0$.
(b) For $p/q = x \in \mathbf{Q}$, the Newton quotient of g at α becomes

$$\left|\frac{g(\alpha) - g(\alpha)}{x - \alpha}\right| = \frac{1}{|p - q\alpha|}.$$

By Exercise 6, §4, of Chapter 1 we know that given $N > 0$ there exists integers p_N, q_N such that

$$\left|\frac{1}{p_N - q_N\alpha}\right| \geq N \quad \text{and} \quad \left|\frac{p_N}{q_N} - \alpha\right| \leq \frac{1}{N},$$

thus g is not differentiable at α.

Exercise III.1.2 *(a) Show that the function $f(x) = |x|$ is not differentiable at 0. (b) Show that the function $g(x) = x|x|$ is differentiable for all x. What is its derivative?*

Solution. (a) For $h > 0$ we have

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1,$$

and if $h < 0$, then

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{-h}{h} = -1,$$

whence f is not differentiable at 0.
(b) If $x > 0$, then $f(x) = x^2$ and if $h > 0$ we get

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} h = 0.$$

If $x < 0$, then $f(x) = -x^2$ and the Newton quotient at 0 tends to 0 as $h \to 0$ with $h < 0$. Thus f is differentiable for all x and for $x > 0$ its derivative is $2x$ and for $x < 0$ its derivative is $-2x$.

Exercise III.1.3 *For a positive integer k, let $f^{(k)}$ denote the k-th derivative of f. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial. Show that for all k,*

$$P^{(k)}(0) = k!a_k.$$

Solution. We prove by induction that for $0 \leq k \leq n$ we have the formula

$$P^{(k)}(x) = k!a_k + \frac{(k+1)!}{1!}a_{k+1}x + \frac{(k+2)!}{2!}a_{k+2}x^2 + \cdots + \frac{(k+n-k)!}{(n-k)!}a_n x^{n-k}.$$

When $k = 0$ the formula holds. Differentiating the above expression we get

$$(k+1)!a_{k+1} + \frac{(k+2)!}{1!}a_{k+2}x + \cdots + \frac{(k+n-k)!}{(n-k-1)!}a_n x^{n-k-1}$$

which is equal to $P^{(k+1)}(x)$, thereby concluding the proof by induction. We immediadely get that $P^{(k)}(0) = k!a_k$ whenever $0 \leq k \leq n$. If $k > n$, then $P^{(k)}$ is identically 0.

Exercise III.1.4 *By induction, obtain a formula for the n-th derivative of a product, i.e. $(fg)^{(n)}$, in terms of lower derivatives $f^{(k)}, g^{(j)}$.*

Solution. We prove by induction that

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} (f)^{(k)}(g)^{(n-k)}.$$

When $n = 1$ the formula yields $(fg)' = f'g + fg'$ which holds. Differentiating the above formula using the product rule and splitting the sum in two we get

$$(fg)^{(n+1)} = \sum_{k=0}^{n} \binom{n}{k} (f)^{(k+1)}(g)^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} (f)^{(k)}(g)^{(n+1-k)}.$$

The change of index $j = k + 1$ in the first sum shows that $(fg)^{(n+1)}$ is

$$\begin{aligned}
&= \sum_{j=1}^{n+1} \binom{n}{j-1} (f)^{(j)}(g)^{(n+1-j)} + \sum_{k=0}^{n} \binom{n}{k} (f)^{(k)}(g)^{(n+1-k)} \\
&= (f)^{(0)}(g)^{(n+1)} + (f)^{(n+1)}(g)^{(0)} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] f^{(k)}g^{(n+1-k)} \\
&= (f)^{(0)}(g)^{(n+1)} + (f)^{(n+1)}(g)^{(0)} + \sum_{k=1}^{n} \binom{n+1}{k} f^{(k)}g^{(n+1-k)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} (f)^{(k)}(g)^{(n+1-k)}.
\end{aligned}$$

The second to last equality follows from Exercise 4, §3, of Chapter 0.

III.2 Mean Value Theorem

Exercise III.2.1 *Let $f(x) = a_n x^n + \cdots + a_0$ be a polynomial with $a_n \neq 0$. Let $c_1 < c_2 < \cdots < c_r$ be numbers such that $f(c_i) = 0$ for $i = 1, \ldots, r$. Show that $r \leq n$. [Hint: Show that f' has at least $r-1$ roots, continue to take the derivatives, and use induction.]*

Solution. Suppose $r > n$. By Lemma 2.2, f' has at least one root in (c_j, c_{j+1}) for all $1 \leq j \leq r-1$. Therefore f' has at least $r-1$ distinct roots. Suppose that for some $1 \leq k \leq n-1$, the function $f^{(k)}$ has at least $r-k$ distinct roots, $c_{k,1} < c_{k,2} < \cdots < c_{k,r-k}$. Then by Lemma 2.2, $f^{(k+1)}$ has at least one root in $(c_{k,j}, c_{k,j+1})$ for all $1 \leq j \leq r-k-1$. Thus $f^{(k+1)}$ has at least $r-(k+1)$ distinct roots. Therefore $f^{(n)}$ has at least $r-n$ roots. But $f^{(n)} = a_n n!$, so $f^{(n)}$ has no roots. This contradiction shows that $r \leq n$.

Exercise III.2.2 *Let f be a function which is twice differentiable. Let $c_1 < c_2 < \cdots < c_r$ be numbers such that $f(c_i) = 0$ for all i. Show that f' has at least $r-1$ zeros (i.e. numbers b such that $f'(b) = 0$).*

Solution. Lemma 2.2 implies that for each $1 \leq j \leq r-1$ there exists numbers d_j such that $c_j < d_j < c_{j+1}$ and $f'(d_j) = 0$. So f' has at least $r-1$ roots.

Exercise III.2.3 *Let $a_1, \ldots, a_n$ be numbers. Determine x so that $\sum_{i=1}^n (a_i - x)^2$ is a minimum.*

Solution. Let $f(x) = \sum_{i=1}^n (a_i - x)^2$. The limits $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$ imply that f has a minimum. The minimum verifies $f'(x) = 0$, which is equivalent to

$$-\sum_{i=1}^{n} 2(a_i - x) = 0.$$

We conclude that f is at a minimum at $x = \sum a_i/n$.

Exercise III.2.4 *Let $f(x) = x^3 + ax^2 + bx + c$ where a, b, c are numbers. Show that there is a number d such that f is convex downward if $x \leq d$ and convex upward if $x \geq d$.*

Solution. The function f'' exists and $f''(x) = 6x + 2a$. Then for all $x \leq d = -a/3$, the function f is convex downward, and for all $x \geq d$, f is convex upward.

Exercise III.2.5 *A function f on an interval is said to satisfy a* **Lipschitz condition** *with* **Lipschitz constant** *C if for all x, y in the interval, we have*

$$|f(x) - f(y)| \leq C|x - y|.$$

Prove that a function whose derivative is bounded on an interval is Lipschitz. In particular, a C^1 function on a closed interval is Lipschitz. Also, note that a Lipschitz function is uniformly continuous. However, the converse if not necessarily true. See Exercise 5 of Chapter IV, §3.

Solution. Let M be a bound for the derivative. Given x and y in the interval, there exists c in (x, y) such that $f(x) - f(y) = f'(c)(x - y)$ which implies

$$|f(x) - f(y)| = |f'(c)||x - y| \leq M|x - y|.$$

3) $f(x) = \sqrt{x}$ is uniformly continuous on $[0,1]$

It has unbounded derivative and is hence not Lipschitz

7) $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$ is not continuous on $[0,1]$

To show f is convex, we must show

$$f((1-v)x + vy) \leq (1-v)f(x) + vf(y) \quad \forall\, x, y. \;\forall\, v \in (0,1)$$

If $x \neq 0$ & $y \neq 0$, holds trivially.

If $x = 0$, $y \neq 0$ we must show $f(vy) \leq (1-v)f(0) + vf(y) \quad \forall\, v \in (0,1)$

This inequality holds, as $f(vy) = 0 \;\forall\, v \in (0,1)$ and $1-v > 0$ and $f(0), f(y) \geq 0$.

If $x = 0, y = 0$, holds immediately

If $x \neq 0, y = 0$, same as case w $x = 0, y \neq 0$.