Exercise 0.2.2 *Show that the equalities of Exercise 1 remain true if the intersection and union signs $\cap$ and $\cup$ are interchanged.*

Solution. We want to show that $S \cup (T \cap T') = (S \cup T) \cap (S \cup T')$. Suppose $x \in S \cup (T \cap T')$, then x belongs to S or T and T'. But since $S \subset (S \cup T) \cap (S \cup T')$ and $(T \cap T') \subset (S \cup T) \cap (S \cup T')$ we must have $x \in (S \cup T) \cap (S \cup T')$. Conversely, if $x \in (S \cup T) \cap (S \cup T')$, then x belongs to $(S \cup T)$ and $(S \cup T')$. If x does not belong to S, then it must lie in T and T', thus lies in $S \cup (T \cap T')$ as was to be shown. The same argument as in Exercise 1 with union and intersection signs interchanged shows that if $T_1, \dots, T_n$ are sets, then

$$S \cup (T_1 \cap \dots \cap T_n) = (S \cup T_1) \cap \dots \cap (S \cup T_n).$$

Exercise 0.2.3 *Let A, B be subsets of a set S. Denote by A^c the complement of A in S. Show that the complement of the intersection is the union of the complements, i.e.*

$$(A \cap B)^c = A^c \cup B^c \quad \textit{and} \quad (A \cup B)^c = A^c \cap B^c.$$

Solution. Suppose $x \in (A \cap B)^c$, so x is not in both A and B, that is $x \notin A$ or $x \notin B$, thus

$$(A \cap B)^c \subset (A^c \cup B^c).$$

Conversely, if $x \in (A^c \cup B^c)$, then $x \notin A$ or $x \notin B$ so certainly, $x \notin A \cap B$, thus $x \in (A \cap B)^c$. Hence

$$(A^c \cup B^c) \subset (A \cap B)^c.$$

For the second formula, suppose $x \in (A \cup B)^c$, then $x \notin A \cup B$, so $x \notin A$ and $x \notin B$, thus $x \in A^c \cap B^c$. Conversely, if $x \notin A$ and $x \notin B$, then $x \notin A \cup B$ so $x \in (A \cup B)^c$.

Exercise 0.2.4 *If X, Y, Z are sets, show that*

$$(X \cup Y) \times Z = (X \times Z) \cup (Y \times Z),$$

$$(X \cap Y) \times Z = (X \times Z) \cap (Y \times Z),$$

Solution. Suppose $(a, b) \in (X \cup Y) \times Z$. Then $b \in Z$ and $a \in X$ or $a \in Y$, so $(a, b) \in (X \times Z)$ or $(a, b) \in (Y \times Z)$, thus $(a, b) \in (X \times Z) \cup (Y \times Z)$. Conversely, if $(a, b) \in (X \times Z) \cup (Y \times Z)$, then $b \in Z$ and $a \in X$ or $a \in Y$. Therefore $(a, b) \in (X \cup Y) \times Z$. This proves the first formula.

For the proof of the second formula, suppose that $(a, b) \in (X \cap Y) \times Z$, then $a \in X \cap Y$ and $b \in Z$. Hence $a \in X, a \in Y$ and $b \in Z$, thus $(a, b) \in (X \times Z)$ and $(a, b) \in (Y \times Z)$. This implies that $(a, b) \in (X \times Z) \cap (Y \times Z)$. Conversely, if $(a, b) \in (X \times Z) \cap (Y \times Z)$, then $(a, b) \in (X \times Z)$ and $(a, b) \in (Y \times Z)$, which implies that $a \in X$, $a \in Y$, and $b \in Z$. Thus $a \in (X \cap Y)$ and $b \in Z$. This implies that $(a, b) \in (X \cap Y) \times Z$ as was to be shown.

Exercise 0.2.5 *Let $f : S \to T$ be a mapping, and let Y, Z be subsets of T. Show that*

$$f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z),$$
$$f^{-1}(Y \cup Z) = f^{-1}(Y) \cup f^{-1}(Z).$$

Solution. If $x \in f^{-1}(Y \cap Z)$, then $f(x) \in Y$ and $f(x) \in Z$, so $x \in f^{-1}(Y) \cap f^{-1}(Z)$. Conversely, if $x \in f^{-1}(Y) \cap f^{-1}(Z)$, then $f(x) \in Y$ and $f(x) \in Z$, so $f(x) \in Y \cap Z$ and therefore $x \in f^{-1}(Y \cap Z)$. This proves the first equality.

For the second equality suppose that $x \in f^{-1}(Y \cup Z)$, then $f(x) \in (Y \cup Z)$, so $f(x) \in Y$ or $f(x) \in Z$ which implies that $x \in f^{-1}(Y) \cup f^{-1}(Z)$. Conversely, if $x \in f^{-1}(Y) \cup f^{-1}(Z)$, then $f(x) \in Y$ or $f(x) \in Z$ which implies that $x \in f^{-1}(Y \cup Z)$.

Exercise 0.2.6 *Let S, T, U be sets, and let $f : S \to T$ and $g : T \to U$ be mappings. (a) If g, f are injective, show that $g \circ f$ is injective. (b) If f, g are surjective, show that $g \circ f$ is surjective.*

Solution. (a) Suppose that $x, y \in S$ and $x \neq y$. Since f and g are injective we have $f(x) \neq f(y)$ and therefore $g(f(x)) \neq g(f(y))$, thus $g \circ f$ is injective.
(b) Since g is surjective, given $y \in U$ there exists $z \in T$ such that $g(z) = y$. Since f is surjective, there exists $x \in S$ such that $f(x) = z$. Then $g(f(x)) = y$, so $g \circ f$ is surjective.

Exercise 0.2.7 *Let S, T be sets and let $f : S \to T$ be a mapping. Show that f is bijective if and only if f has an inverse mapping.*

Solution. Suppose that f is bijective. Given any $y \in T$ there exists $x \in S$ such that $f(x) = y$ because f is surjective, and this x is unique because f is injective. Define a mapping $g : T \to S$ by $g(y) = x$, where x is the unique element of S such that $f(x) = y$. Then by construction we have $f \circ g = \mathrm{id}_T$ and $g \circ f = \mathrm{id}_S$.

Conversely, suppose that f has an inverse mapping g. Then given any $y \in T$ we have $f(g(y)) = y$ so f is surjective. If $x, x' \in S$ and $x \neq x'$, then $g(f(x)) \neq g(f(x'))$ which implies that $f(x) \neq f(x')$. Thus f is injective.

0.3 Natural Numbers and Induction

(In the exercises you may use the standard properties of numbers concerning addition, multiplication, and division.)

Exercise 0.3.1 *Prove the following statements for all positive integers.*

(a) $1 + 3 + 5 + \dots + (2n - 1) = n^2$.
(b) $1^2 + 2^2 + 3^2 + \dots + n^2 = n(n + 1)(2n + 1)/6$.
(c) $1^3 + 2^3 + 3^3 + \dots + n^3 = [n(n + 1)/2]^2$.

Solution. (a) For $n = 1$ we certainly have $1 = 1$. Assume the formula is true for an integer $n \geq 1$. Then

$$1+3+5+\cdots+(2n-1)+(2(n+1)-1) = n^2+2(n+1)-1 = n^2+2n+1 = (n+1)^2.$$

(b) For $n = 1$ we certainly have $1^2 = (1 \cdot 2 \cdot 3)/6$. Assume the formula is true for some integer $n \geq 1$. Then

$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 &= \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\ &= \frac{(n+1)(2n^2+7n+6)}{6} \\ &= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}. \end{aligned}$$

(c) For $n = 1$ we have $1^3 = (2/2)^3$. Assume the formula is true for an integer $n \geq 1$. Then

$$\begin{aligned} 1^3 + 2^3 + 3^3 + \cdots + n^3 + (n+1)^3 &= \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \\ &= \left[\frac{(n+1)(n+2)}{2}\right]^2. \end{aligned}$$

Exercise 0.3.2 *Prove that for all numbers $x \neq 1$,*

$$(1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n}) = \frac{1-x^{2^{n+1}}}{1-x}.$$

Solution. The formula is true for $n = 0$ because $(1+x)(1-x) = 1-x^2$. Assume the formula is true for an integer $n \geq 0$, then

$$\begin{aligned} (1+x)(1+x^2)(1+x^4)\cdots(1+x^{2^n})(1+x^{2^{n+1}}) &= \frac{1-x^{2^{n+1}}}{1-x}(1+x^{2^{n+1}}) \\ &= \frac{1-\left(x^{2^{n+1}}\right)^2}{1-x} \\ &= \frac{1-x^{2^{n+2}}}{1-x}. \end{aligned}$$

Exercise 0.3.3 *Let $f\colon \mathbf{N} \to \mathbf{N}$ be a mapping such that $f(xy) = f(x)+f(y)$ for all x, y. Show that $f(a^n) = nf(a)$ for all $n \in \mathbf{N}$.*

Solution. The formula is true for $n = 1$ because $f(a^1) = 1 \cdot f(a)$. Suppose the formula is true for an integer $n \geq 1$. Then

$$f(a^{n+1}) = f(a^n a) = f(a^n) + f(a) = nf(a) + f(a) = (n+1)f(a),$$

as was to be shown.

Exercise 0.3.4 *Let $\binom{n}{k}$ denote the binomial coefficient,*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where n, k are integers ≥ 0, $0 \leq k \leq n$, and $0!$ is defined to be 1. Also $n!$ is defined to be the product $1 \cdot 2 \cdot 3 \cdots n$. Prove the following assertions.

(a) $\binom{n}{k} = \binom{n}{n-k}$ *(b)* $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ *(for $k > 0$)*

Solution. (a) This result follows from

$$\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}.$$

(b) We simply have

$$\begin{aligned} \binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{(n-k)!k!} \\ &= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} = \frac{n!(n+1)}{k!(n+1-k)!} \\ &= \binom{n+1}{k}. \end{aligned}$$

Exercise 0.3.5 *Prove by induction that*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Solution. For $n = 1$ the formula is true. Suppose that the formula is true for an integer $n \geq 1$. Then

$$\begin{aligned} (x+y)^{n+1} &= (x+y)\left[\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\right] \\ &= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\ &= \sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \end{aligned}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + \binom{n}{n} x^{n+1} + y^{n+1}$$
$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}$$

as was to be shown. The second to last identity follows from the previous exercise.

Exercise 0.3.6 *Prove that*

$$\left(1+\frac{1}{1}\right)^1 \left(1+\frac{1}{2}\right)^2 \cdots \left(1+\frac{1}{n-1}\right)^{n-1} = \frac{n^{n-1}}{(n-1)!}.$$

Find and prove a similar formula for the product of the terms $(1+1/k)^{k+1}$ *taken for* $k = 1, \ldots, n-1$.

Solution. For $n = 2$ the formula is true because $2 = 2^{2-1}/(2-1)!$. Suppose the formula is true for an integer $n \geq 2$, then

$$\left(1+\frac{1}{1}\right)^1 \cdots \left(1+\frac{1}{n-1}\right)^{n-1} \left(1+\frac{1}{n}\right)^n = \frac{n^{n-1}}{(n-1)!}\left(\frac{n+1}{n}\right)^n = \frac{(n+1)^n}{n!}$$

as was to be shown. The product of the terms $(1+1/k)^{k+1}$ taken for $k = 1, \ldots, n-1$ is given by the formula

$$\left(1+\frac{1}{1}\right)^2 \cdots \left(1+\frac{1}{n-1}\right)^n = \frac{n^n}{(n-1)!}.$$

The proof is also by induction.

0.4 Denumerable Sets

Exercise 0.4.1 *Let* F *be a finite non-empty set. Show that there is a surjective mapping of* $\mathbf{Z}^+$ *onto* F.

Solution. By definition the set F has n elements for some integer $n \geq 1$. There exists a bijection g between F and J_n. Define $f : \mathbf{Z}^+ \to F$ by

$$f(k) = \begin{cases} g(k) & \text{if } k \in J_n, \\ g(n) & \text{if } k > n. \end{cases}$$

The mapping f is surjective.

Exercise 0.4.2 *How many maps are there which are defined on a set of numbers* $\{1, 2, 3\}$ *and whose values are in the set of integers* n *with* $1 \leq n \leq 10$?

Solution. There are 10^3 maps. See the next exercise.

Exercise 0.4.3 *Let* E *be a set with* m *elements and* F *a set with* n *elements. How many maps are there defined on* E *and with values in* F? *[Hint: Suppose first that* E *has one element. Next use induction on* m, *keeping* n *fixed.]*

Solution. We prove by induction that there are n^m maps defined on E with values in F

Suppose $m = 1$. To the single element in E we can assign n values in F. Suppose the induction statement is true for an integer $m \geq 1$. Suppose that E has $m+1$ elements. Choose $x \in E$. To x we can associate n elements of F. For each such association there is n^m maps defined on $E - \{x\}$ with values in F. So there is a total of $n \times n^m = n^{m+1}$ maps defined on E with values in F.

Exercise 0.4.4 *If* S, T, S', T' *are sets, and there is a bijection between* S *and* S', T, *and* T', *describe a natural bijection between* $S \times T$ *and* $S' \times T'$. *Such a bijection has been used implicitly in some proofs.*

Solution. Given the bijections $f : S \to S'$ and $g : T \to T'$ define a mapping $h : S \times T \to S' \times T'$ by

$$h(x, y) = (f(x), g(y)).$$

Given any $(x', y') \in S' \times T'$ there exists $x \in S$ and $y \in T$ such that $f(x) = x'$ and $g(y) = y'$. Then $h(x, y) = (x', y')$, so h is surjective. The map h is injective because if $h(x_1, y_1) = h(x_2, y_2)$, then $f(x_1) = f(x_2)$ and $g(y_1) = g(y_2)$, so $x_1 = x_2$ and $y_1 = y_2$ because both f and g are injective.

0.5 Equivalence Relations

Exercise 0.5.1 *Let* T *be a subset of* $\mathbf{Z}$ *having the property that if* $m, n \in T$, *then* $m+n$ *and* $-n$ *are in* T. *For* $x, y \in \mathbf{Z}$ *define* $x \equiv y$ *if* $x - y \in T$. *Show that this is an equivalence relation.*

Solution. Suppose that T is non-empty, otherwise there is nothing to prove. The element 0 belongs to T because if $n \in T$, then $-n$ and $0 = n - n$ belongs to T. Therefore $x \equiv x$ for all x. Since $y - x = -(x - y)$ we see that $x \equiv y$ implies $y \equiv x$. Finally, if $x \equiv y$ and $y \equiv z$, then $x \equiv z$ because $x - z = (x - y) + (y - z)$ so $x - z \in T$.

Exercise 0.5.2 *Let* $S = \mathbf{Z}$ *be the set of integers. Define the relation* $x \equiv y$ *for* $x, y \in \mathbf{Z}$ *to mean that* $x - y$ *is divisible by 3. Show that this is an equivalence relation.*

Solution. Distributivity and commutativity imply

$$\begin{aligned}(x+y)(x+y) &= (x+y)x+(x+y)y \\ &= x^2+yx+xy+y^2 \\ &= x^2+2xy+y^2.\end{aligned}$$

Similarly,

$$\begin{aligned}(x+y)(x-y) &= (x+y)x-(x+y)y \\ &= x^2+yx-xy-y^2 \\ &= x^2-y^2.\end{aligned}$$

I.2 Ordering Axioms

Exercise I.2.1 *If $0 < a < b$, show that $a^2 < b^2$. Prove by induction that $a^n < b^n$ for all positive integers n.*

Solution. The axioms imply $aa < ba$ and $ab < bb$, so by transitivity (IN 1.) we have $a^2 < b^2$.

The general inequality is true when $n = 1$. Suppose the inequality is true for some integer $n \geq 1$. Then by assumption $a^n < b^n$ and since a and b are both > 0 with $a < b$ we find that

$$a^n a < a^n b \quad \text{and} \quad a^n b < b^n b.$$

Therefore $a^{n+1} < b^{n+1}$, as was to be shown.

Exercise I.2.2 *(a) Prove that $x \leq |x|$ for all real x. (b) If $a, b \geq 0$ and $a \leq b$, and if $\sqrt{a}, \sqrt{b}$ exist, show that $\sqrt{a} \leq \sqrt{b}$.*

Solution. (a) If $x \geq 0$, then $|x| = x$ so $x \leq |x|$. If $x \leq 0$, then $x \leq 0 \leq |x|$ and we get $x \leq |x|$.
(b) Suppose that $\sqrt{b} < \sqrt{a}$. Then by Exercise 1 we know that $(\sqrt{b})^2 < (\sqrt{a})^2$, whence $b < a$, which contradicts the assumption that $a \leq b$.

Exercise I.2.3 *Let $a \geq 0$. For each positive integer n, define $a^{1/n}$ to be a number x such that $x^n = a$, and $x \geq 0$. Show that such a number x, if it exists, is uniquely determined. Show that if $0 < a < b$, then $a^{1/n} < b^{1/n}$ (assuming that the n-th roots exist).*

Solution. If $a = 0$ we must have $a^{1/n} = 0$ for otherwise we get a contradiction with Exercise 2 of §1. Suppose that $a > 0$ and that x exists. Then we must have $x \neq 0$. If $x^n = y^n = a$ and $x, y > 0$, then $x = y$. Indeed, if $x < y$, then $x^n < y^n$ (Exercise 1) which is a contradiction. Similarly we cannot have $y < x$.

Now suppose that $0 < a < b$ and $b^{1/n} \leq a^{1/n}$. Then arguing like in Exercise 1 we find that $(b^{1/n})^n \leq (a^{1/n})^n$ so $b \leq a$ which is a contradiction. So $a^{1/n} < b^{1/n}$ and we are done.

Exercise I.2.4 *Prove the following inequalities for $x, y \in \mathbf{R}$.*

$$\begin{aligned}|x-y| &\geq |x|-|y|, \\ |x-y| &\geq |y|-|x|, \\ |x| &\leq |x+y|+|y|.\end{aligned}$$

Solution. All three inequalities are simple consequences of the triangle inequality. For the first we have

$$|x| = |x-y+y| \leq |x-y|+|y|.$$

The second inequality follows from

$$|y| = |y-x+x| \leq |y-x|+|x|.$$

Finally, the third inequality follows from

$$|x| = |x+y-y| \leq |x+y|+|y|.$$

Exercise I.2.5 *If x, y are numbers ≥ 0 show that*

$$\sqrt{xy} \leq \frac{x+y}{2}.$$

Solution. The inequality follows from the fact that

$$0 \leq (\sqrt{x}-\sqrt{y})^2 = x - 2\sqrt{xy} + y.$$

Exercise I.2.6 *Let b, ϵ be numbers and $\epsilon > 0$. Show that a number x satisfies the condition $|x-b| < \epsilon$ if and only if*

$$b-\epsilon < x < b+\epsilon.$$

Solution. Suppose that $|x-b| < \epsilon$. If $b \leq x$, then $0 \leq |x-b| = x-b < \epsilon$, so $x < b+\epsilon$. If $x \leq b$, then $0 \leq |x-b| = b-x < \epsilon$, so $b-\epsilon < x$.

Conversely, if $b-\epsilon < x < b+\epsilon$, then $-\epsilon < x-b < \epsilon$ so $|x-b| < \epsilon$.

Exercise I.2.7 *Notation as in Exercise 6, show that there are precisely two numbers x satisfying the condition $|x-b| = \epsilon$.*

Solution. Since $\epsilon > 0$ we must have $x \neq b$. If $x > b$, the equation $|x-b| = \epsilon$ is equivalent to $x - b = \epsilon$ which has a unique solution namely, $x = b+\epsilon$. If we have $x < b$, then $|x-b| = \epsilon$ is equivalent to $b - x = \epsilon$ which has a unique solution, namely $x = b-\epsilon$. So $|x-b| = \epsilon$ has exactly two solutions, $b+\epsilon$ and $b-\epsilon$.

Exercise I.2.8 *Determine all intervals of numbers satisfying the following equalities and inequalities:*
(a) $x + |x-2| = 1 + |x|$. *(b)* $|x-3| + |x-1| < 4$.

Solution. (a) Suppose $x \geq 2$. Then the equation is equivalent to $x + x - 2 = 1 + x$ which has a unique solution $x = 3$. If $0 \leq x \leq 2$ we are reduced to $x + 2 - x = 1 + x$ which has only one solution given by $x = 1$. If $x \leq 0$, then $x + 2 - x = 1 - x$ has a unique solution $x = -1$. So the set of solution to $x + |x-2| = 1 + |x|$ is $S = \{-1, 1, 3\}$.
(b) Separating the cases, $3 \leq x$, $1 \leq x \leq 3$, and $x \leq 1$ we find that the interval solution is $S = (0, 4)$.

Exercise I.2.9 *Prove: If* x, y, ϵ *are numbers and* $\epsilon > 0$, *and if* $|x - y| < \epsilon$, *then*

$$|x| < |y| + \epsilon, \quad \text{and} \quad |y| < |x| + \epsilon.$$

Also,

$$|x| > |y| - \epsilon, \quad \text{and} \quad |y| > |x| - \epsilon.$$

Solution. Using the first inequality of Exercise 4 we get

$$|x| \leq |x - y| + |y| < \epsilon + |y|.$$

By the second inequality of Exercise 4 we get

$$|y| \leq |x - y| + |x| < \epsilon + |x|,$$

so $|y| < \epsilon + |x|$.

Exercise I.2.10 *Define the* **distance** $d(x, y)$ *between two numbers* x, y *to be* $|x - y|$. *Show that the distance satisfies the following properties:* $d(x, y) = d(y, x)$; $d(x, y) = 0$ *if and only if* $x = y$; *and for all* x, y, z *we have*

$$d(x, y) \leq d(x, z) + d(z, y).$$

Solution. We have

$$d(x, y) = |x - y| = |-(y - x)| = |y - x| = d(y, x).$$

Cleary, $x = y$ implies $d(x, y) = 0$ conversely, if $d(x, y) = 0$, then $|x - y| = 0$ so by the standard property of the absolute value (i.e. $|a| = 0$ if and only if $a = 0$) we conclude that $x - y = 0$, thus $x = y$. The last property follows from the triangle inequality for the absolute value:

$$d(x, y) = |x - y| = |x - z + z - y| \leq |x - z| + |z - y| = d(x, z) + d(z, y).$$

Exercise I.2.11 *Prove by induction that if* $x_1, \ldots, x_n$ *are numbers, then*

$$|x_1 + \cdots + x_n| \leq |x_1| + \cdots + |x_n|.$$

Solution. If $n = 1$ the inequality is obviously true. Suppose that the inequality is true for some integer $n \geq 1$. Then, by the triangle inequality and the induction hypothesis we obtain

$$|x_1 + \cdots + x_n + x_{n+1}| \leq |x_1 + \cdots + x_n| + |x_{n+1}| \leq |x_1| + \cdots + |x_n| + |x_{n+1}|,$$

as was to be shown.

I.3 Integers and Rational Numbers

Exercise I.3.1 *Prove that the sum of a rational number and an irrational number is always irrational.*

Solution. If not, then for some rational numbers x, y and some $\alpha \notin \mathbf{Q}$ we have $x + \alpha = y$. Then $\alpha = y - x$, but the difference of two rational numbers is rational, so $\alpha \in \mathbf{Q}$, which is a contradiction.

Exercise I.3.2 *Assume that* $\sqrt{2}$ *exists, and let* $\alpha = \sqrt{2}$. *Prove that there exists a number* $c > 0$ *such that for all integers* q, p, *and* $q \neq 0$ *we have*

$$|q\alpha - p| > \frac{c}{q}.$$

[Note: The same c should work for all q, p. *Try rationalizing* $q\alpha - p$, *i.e. take the product* $(q\alpha - p)(-q\alpha - p)$, *show that it is an integer, so that its absolute value is* ≥ 1. *Estimate* $q\alpha + p$.*]*

Solution. We may assume without loss of generality that $q > 0$. Let $a = 2$ in the solution of Exercise 4.

Exercise I.3.3 *Prove that* $\sqrt{3}$ *is irrational.*

Solution. Suppose that $\sqrt{3}$ is rational and write $\sqrt{3} = p/q$, where the fraction is in lowest form. Then $3q^2 = p^2$. If q is even, then $3q^2$ is even, which implies that p is even. This is a contradiction because the fraction p/q is in lowest form.
If q is odd, then $3q^2$ is odd, thus p must be odd. Suppose $q = 2n + 1$ and $p = 2m + 1$. Then we can rewrite $3q^2 = p^2$ as

$$12n^2 + 12n + 3 = 4m^2 + 4m + 1$$

which is equivalent to

$$6n^2 + 6n + 1 = 2m^2 + 2m.$$

The left-hand side of the above equality is odd and the right hand side is even. This contradiction shows that q cannot be odd and concludes the proof that $\sqrt{3}$ is not rational.

Exercise I.3.4 *Let a be a positive integer such that $\sqrt{a}$ is irrational. Let $\alpha = \sqrt{a}$. Show that there exists a number $c > 0$ such that for all integers p, q with $q > 0$ we have*

$$|q\alpha - p| > c/q.$$

Solution. We follow the suggestion given in Exercise 2. We have

$$(q\alpha - p)(-q\alpha - p) = -q^2\alpha^2 + p^2 = -q^2a + p^2 \in \mathbf{Z}^* = \mathbf{Z} - \{0\},$$

because α is irrational and $\alpha^2 = a$ is an integer. So the absolute value of the left-hand side is ≥ 1 which gives

$$|q\alpha - p| \geq \frac{1}{|q\alpha + p|}.$$

Let c be a number such that $0 < c < \min\{|\alpha|, 1/(3|\alpha|)\}$. We consider two cases.

Suppose that $|\alpha - p/q| < |\alpha|$, then

$$\left|\alpha + \frac{p}{q}\right| \leq |2\alpha| + \left|-\alpha + \frac{p}{q}\right| < 3|\alpha|.$$

Therefore

$$|q\alpha - p| \geq \frac{1}{|q\alpha + p|} > \frac{1}{3|\alpha|q} > \frac{c}{q}.$$

If $|\alpha - p/q| \geq |\alpha|$, then

$$|q\alpha - p| \geq q|\alpha| > \frac{c}{q}.$$

This concludes the exercise.

Exercise I.3.5 *Prove: Given a non-empty set of integers S which is bounded from below (i.e. there is some integer m such that $m < x$ for all $x \in S$), then S has a least element, that is an integer n such that $n \in S$ and $n \leq x$ for all $x \in S$. [Hint: Consider the set of all integers $x - m$ with $x \in S$, this being a set of positive integers. Show that if k is its least element, then $m + k$ is the least element of S.]*

Solution. Let $T = \{y \in \mathbf{Z} : y = x - m$ for some $x \in S\}$. The set T is non-empty and $T \subset \mathbf{Z}^+$. The well-ordering axiom implies that T has a least element k. Then for some $x_0 \in S$ we have $k = x_0 - m$ so $x_0 = k + m$. Clearly for all $x \in S$ we have

$$x - x_0 = x - m - (x_0 - m) = x - m - k \geq 0.$$

I.4 The Completeness Axiom

Exercise I.4.1 *In Proposition 4.3, show that one can always select the rational number a such that $a \neq z$ (in case z itself is rational). [Hint: If z is rational, consider $z + 1/n$.]*

Solution. If z is irrational, then there is no problem. If z is rational, let $a = z + 1/n \in \mathbf{Q}$, where $1/n < \epsilon$. Then $|z - a| \leq 1/n < \epsilon$.

Exercise I.4.2 *Prove: Let w be a rational number. Given $\epsilon > 0$, there exists an irrational number y such that $|y - w| < \epsilon$.*

Solution. Choose $z \in \mathbf{Q}$ such that $|(w/\sqrt{2}) - z| < \epsilon/\sqrt{2}$. Then $y = z\sqrt{2} \notin \mathbf{Q}$, and $|y - w| < \epsilon$.

Exercise I.4.3 *Prove: Given a number z, there exists an integer n such that $n \leq z < n + 1$. This integer is usually denoted by $[z]$.*

Solution. Let $S = \{n \in \mathbf{Z}$ such that $z - 1 < n\}$ which is non-empty. Then $n_0 = \inf(S)$ exists by Exercise 5 of the preceding section and $n_0 \in S$. Hence $z - 1 < n_0$. We cannot have $z - 1 < n_0 - 1$ because $n_0 = \inf(S)$, thus $z - 1 \geq n_0 - 1$, which implies $z \geq n_0$. Putting everything together we see that $n_0 \leq z < n_0 + 1$.

Exercise I.4.4 *Let $x, y \in \mathbf{R}$. Define $x \equiv y$ if $x - y$ is an integer. Prove:*
(a) This defines an equivalence relation in $\mathbf{R}$.
(b) If $x \equiv y$ and k is an integer, then $kx \equiv ky$.
(c) If $x_1 \equiv y_1$ and $x_2 \equiv y_2$, then $x_1 + x_2 \equiv y_1 + y_2$.
(d) Given a number $x \in \mathbf{R}$, there exists a unique number $\overline{x}$ such that $0 \leq \overline{x} < 1$ and such that $\overline{x} \equiv x$ (in other words, $x - \overline{x}$ is an integer). Show that $\overline{x} = x - [x]$, where the bracket is that of Exercise 3.

Solution. (a) Since 0 is an integer, $x \equiv x$ for all x. If $x \equiv y$, then $y \equiv x$ because $y - x$ is an integer whenever $x - y$ is an integer. Finally if $x \equiv y$ and $y \equiv x$, then $x \equiv z$ because $x - z = x - y + y - z$.
(b) The result follows from the fact that $kx - ky = k(x - y)$.
(c) Immediate from the fact that $(x_1+x_2)-(y_1+y_2) = (x_1-y_1)+(x_2-y_2)$.
(d) By Exercise 3, we know that given $x \in \mathbf{R}$ there exists an integer n such that $n \leq x < n + 1$. Let $\overline{x} = x - n = x - [x]$. Then $0 \leq \overline{x} < 1$, thereby proving existence. For uniqueness suppose that there exists two numbers a and b such that $0 \leq a, b < 1$ and $a \equiv x$ and $b \equiv x$. Then by (b) and (c) $a - b \equiv 0$ so $a - b$ is an integer. But $0 \leq a, b < 1$, hence $-1 < a - b < 1$ which implies that $a - b = 0$ as was to be shown.

Exercise I.4.5 *Denote the number $\overline{x}$ of Exercise 4 by $R(x)$. Show that if x, y are numbers, and $R(x) + R(y) < 1$, then $R(x + y) = R(x) + R(y)$. In general, show that*

$$R(x + y) \leq R(x) + R(y).$$