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Analysis I
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Day 9 — Summary — Taylor Series

46. Big oh and Little oh notation:

    (a) $f(x) = o(g(x))$ as $x \to x_0$ means that $f(x)/g(x) \to 0$ as $x \to x_0$

    (b) $f(x) = O(g(x))$ as $x \to x_0$ means that there exists C such that $|f(x)| \leq Cg(x)$

47. A Taylor series is a local approximation of a function, and it is obtained by matching the value and a given number of derivatives of that function at a particular point.

48. The nth order Taylor series of $f(x)$ about $x = a$ is given by

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$$

49. The nth Taylor remainder term is

$$R_n(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n \right).$$

50. The nth order Taylor series is accurate to the $n+1$st order in the neighborhood of the point of expansion. The constant factor of the error term is controlled by the maximum value of the $n+1$st derivative of the function.

    If $f \in C^{n+1}$ in a neighborhood of a, then $R_n(x) = O(|x-a|^{n+1})$ as $x \to a$. More precisely,

$$R_n(x) \leq \max |f^{(n+1)}| \cdot \frac{|x-a|^n}{n!}.$$

    The max is taken over the neighborhood and the inequality holds for all points in the neighborhood.