3 September 2015
Analysis I
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Day 4 — Summary — Squeeze theorem, limits and infinity, continuity and extrema

23. Squeeze theorem: Suppose $f(x) \leq g(x) \leq h(x)$ for x sufficiently close to a. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x)$ exists and is also equal to L.

24. $\lim_{x\to\infty} f(x) = L$ if for all ε, there exists a C such that $x > C \Rightarrow |f(x) - L| < \varepsilon$.
Corresponding definitions for $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = \infty$.

25. If $\lim_{x\to\infty} f(x) = L > 0$ and $\lim_{x\to\infty} g(x) = \infty$, then $\lim_{x\to\infty}(fg)(x) = \infty$.

26. Extreme value theorem: A continuous function over a closed bounded interval achieves its maximum and minimum.

27. Let $a > 1, k \in \mathbb{N}$. $\lim_{n\to\infty} a^n/n^k = \infty$.

21) Uniform continuity:

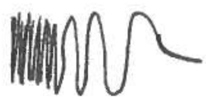
A function can not change by more than ϵ in arb. small distance

Examples:

$f(x) = \sqrt{1-x^2}$ on $[-1, 1]$

Non examples

$f(x) = x^2$ on $\mathbb{R}$



$f(x) = \sin(\frac{1}{x})$ on $(0, \infty)$

Activity: Uniformly continuos or not

a) $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ on $[-1, 1]$

b) $f(x) = \sin(1+x^2)$ on $\mathbb{R}$

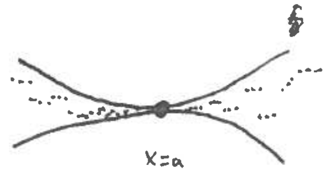
c) $f(x) = \sin\left(\frac{1}{1+x^2}\right)$ on $\mathbb{R}$

d) Let $f(x) = f(x+2\pi)$.
Let $f \in C(\mathbb{R})$.

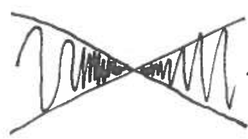
23) Squeeze Theorem

Idea: If a function is sandwiched between two functions that share a limit, then all three share the limit.

Picture:



— this function is not continuous, but it has a limit as $x \to a$

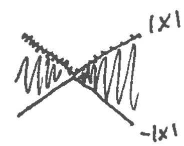


this function gets arbitrarily steep, but still has a limit

Application:

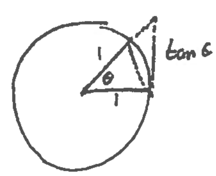
$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\frac{1}{x} & \text{if } x \neq 0 \end{cases}$$ is continuous at $x = 0$.

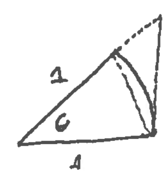
It is squeezed between $|x|$ and $-|x|$.
Hence it has limit 0



Application: Show $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Pf:





Area of small triangle $\leq$ area of sector $\leq$ area of big triangle

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \tan\theta$$

$$\Rightarrow \quad 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta.$$

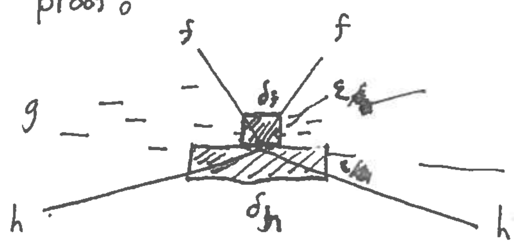
Squeezed $\frac{\sin\theta}{\theta}$ between 1 & $\cos\theta$ which $\to 1$ as $\theta \to 0$.

Formal Statement:

If $f(x) \le g(x) \le h(x)$ for x suff. near a

and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

Gist of proof:

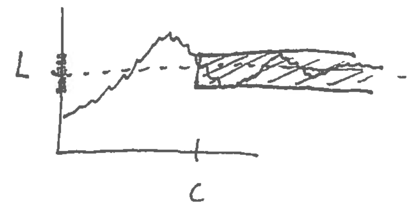


For a given ε, choose the smaller of the δ_f & δ_h corresponding to ε error from the limit. Because g squeezed between, its error is $\le \varepsilon$ within δ_f & δ_h.

24) $\lim_{x \to \infty} f(x) = L$

Idea: f ~~gets~~ is arbitrarily close to L for sufficiently large x

Picture:



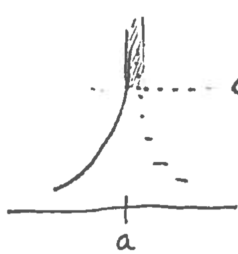
Some tail fits entirely in box of height ε for any ε.

Formal statement: $\forall \varepsilon \; \exists C$ st $\forall x \geq C \quad |f(x) - L| \leq \varepsilon$

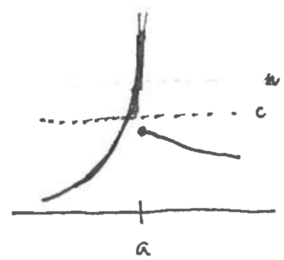
$\lim_{x \to a} f(x) = \infty$

Idea: f ~~gets~~ is arbitrarily large for x sufficiently near a

Picture:



$\lim = \infty$



$\lim \neq \infty$

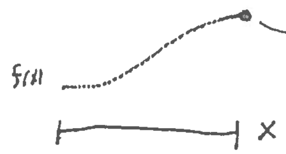
– ~~Set of~~ There are points arbitrarily near x=c such that f is not larger than C

Formal statement: $\forall C \; \exists \delta$ st $\forall \; |x-a| < \delta \quad f(x) \geq C$.

26) Extreme Value Theorem

A continuous function on a closed bounded interval achieves its max and min

Picture:



if function gets very close to this y value, it must achieve it.

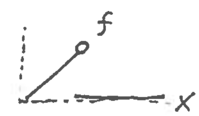
Consequence of theorem: $\min_{x \in [a,b]} f(x)$ – this minimization problem has a minimizer (an x that adopts the minimal value)

Non Examples:

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x \geq 1 \end{cases}$$

never achieves max on $[0,2]$

it is not continuous



$f(x) = x$ on $0 < x < 1$ never achieves max on $(0,1)$

Note: when max is not achieved we dont use word "max" we use "sup"

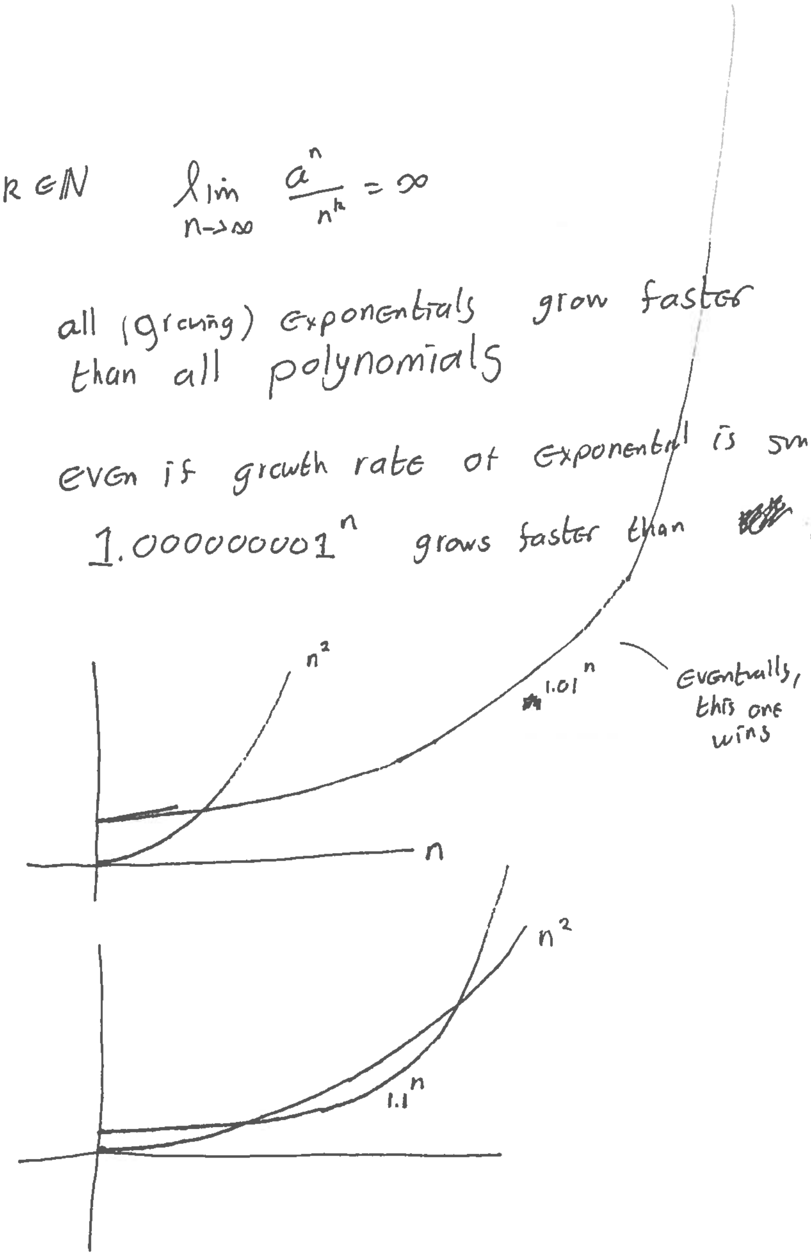
Proof:
- Take subsequence such that f approaches supremum
- By Bolzano Weierstrass, $\exists$ conv. subseq
- Defn of continuity gives that max achieved.

5) $a > 1$, $k \in \mathbb{N}$ $\lim_{n \to \infty} \frac{a^n}{n^k} = \infty$

Intuition: all (growing) exponentials grow faster than all polynomials

~~Picture:~~ Even if growth rate of exponential is small 1.000000001^n grows faster than $n^{(10^{10})}$

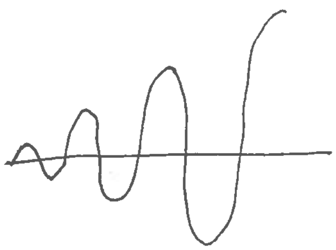
Picture:



Significance: Theoretical Computer science is based on trying to finding algorithms that run in polynomial time. "Dumb" methods like exhaustive search take exponentially many operations. ~~Polyno~~ Algorithms that are polynomial time will outperform them for large problem sizes.

Proof: $\lim \frac{(1+b)^n}{n^k}$ Expand $(1+b)^n$ using binomial theorem. Keep one term to get bound.

Activity: Draw and write down a smooth function that is not bounded but does not converge to ∞ as

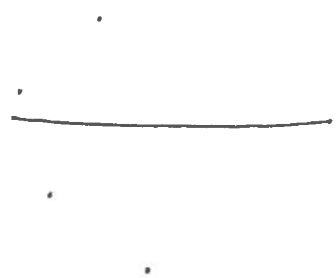


$f(x) = x\sin$

Activity: Draw and specify a sequence x_n that is never 0. yet converges to 0 yet $\frac{1}{x_n}$ does not have ∞ as a limit

x_n:

$\frac{1}{x_n}$:



$x_n = (-1)^n/n$

Questions: If I have a sequence of #'s in $\mathbb{R}$, is it true that there is a subsequence that converges (to a real number or $+\infty$ or $-\infty$) ?

Yes. Either bounded or unbdd from above or unbdd from below.

Activity: What does it mean that a set S is unbound from above

$\nexists\, C$ such that $x \leqslant C \ \forall x \in S$

$\forall\, C \ \exists\, x \in S$ s.t. $x > C$.