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Analysis I
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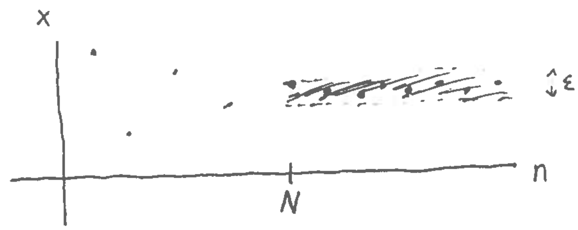
Day 2— Summary — Cauchy sequences, Bolzano-Weierstrass, limsup and liminf

9. The sequence $\{x_n\}$ is Cauchy if $\forall\ \varepsilon > 0$, there exists N such that $m, n \geq N \Rightarrow |x_m - x_n| < \varepsilon$.
10. $\mathbb{R}$ is complete: If $\{x_n\}$ is a Cauchy sequence of $\mathbb{R}$, then $\{x_n\}$ converges to an element of $\mathbb{R}$.
11. Let $x = \{x_n\}$ be a sequence. A subsequence of x is obtained by keeping (in order) an infinite number of the items x_n and discarding the rest. Two ways to denote a subsequence are $x_{(n)}$ and x_{n_k}.
12. Let $\{x_n\}$ be a sequence. The number x is an accumulation point (or point of accumulation) of the sequence if $\forall \varepsilon$ there are infinitely many n such that $|x_n - x| < \varepsilon$.
13. Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.
14. (a) $\limsup\{x_n\}$ is defined as supremum of the accumulation points of $\{x_n\}$. An alternative way to think about it is through $\limsup\{x_n\} = \lim_{n->\infty} \sup_{m \geq n} x_m$.
    (b) $\liminf\{x_n\}$ is defined analogously.

9 $\{X_n\}$ is Cauchy if $\forall \varepsilon \ \exists N$ st $\forall n,m \geq N$ $|X_n - X_m| < \varepsilon$

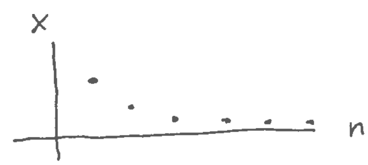
Visually:



Visual nonexample:

Cant put tail in strip of height

Example: $X_n = \frac{1}{n}$ is Cauchy



To prove, we need way of getting N from ε.

Fix ε. Let N be such that $\frac{2}{N} < \varepsilon$.

If, $n, m \geq N$ then $|X_n - X_m| \leq |X_n| + |X_m| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{N} + \frac{1}{N} = \frac{2}{N} < \varepsilon$

Nonexample: Prove $X_n = (-1)^n$ is not Cauchy.

Need to show $\exists \varepsilon$ such that $\forall N$ $\exists n, m \geq N$ with $|X_n - X_m| \geq \varepsilon$

Let $\varepsilon = 1/2$. and fix N. If $n = N+1$, $m = N$ then $|X_n - X_m| = 2 > \varepsilon$.

Another way to think of Cauchy sequences in $\mathbb{R}$

for all ε there is some tail of the sequence such that the supremum and infimum over the tail are within ε.

What is not so great of this view?

Doesnt generalize to other spaces lacking least upper bound property

Why do~~nt~~ we care about Cauchy-ness? If we ultimately want convergence,
why not just ~~say~~ use defn: X_n converges if $\exists\, L$ such that $\lim_{n\to\infty} X_n = L$.

Cauchy criterion is a test for convergence.
Only involves sequence items themselves. L may not even live in the same space.

10) If X_n is Cauchy sequence in $\mathbb{R}$, X_n converges

Example: $X_n = \frac{1}{n}$ is Cauchy and converges to 0

Proof: (Sketch) Consider the tails of X_n corresponding to $n \geq N$

$b_n = \inf_{k \geq N} X_k$ is a monotonic increasing seq.

Cauchy $\Rightarrow$ bounded above and below $\Rightarrow$ $\inf_{n \geq N} X_n \to b$

We will show $X_n \to b$. Fix ε

~~Beyond N_1, $\inf_{n \geq N_1} X_n \geq b - \frac{\varepsilon}{3}$ so $X_n \geq b - \frac{\varepsilon}{3}$ $\forall n \geq N_1$~~

~~Beyond N_2, $|X_n - X_m| \leq \varepsilon/3$~~

Beyond N_1, $|b_n - b| < \varepsilon/3$ (limit of b_n)

Beyond N_2, $|X_n - X_m| < \varepsilon/3$ (Cauchy)

Beyond any N, there is $m \geq n$ st $|b_n - X_m| < \varepsilon/3$ (dfn of b_n)

So $\forall n \geq N = \max(N_1, N_2)$

$$|X_n - b| \leq |X_n - X_m| + |-b_n + X_m| + |b_n - b|$$
$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Visually

Sequence of smallest items with a tail is monotone increasing

#11 $X = \{X_n\} = \{X_1, X_2, X_3, X_4, \cdots\}$

A subsequence is given by X_{n_k} where $n_{k+1} > n_k$.

That is, keep infinitely many X_n, discard the rest.

Notation $X_{(n)}$ or X_{n_k}

Example: • $\{0, 1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, 0, \frac{1}{5}, \cdots\}$

has subseq $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots\}$

• $\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}$

has subseq $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \cdots\}$

Nonexamples of a Cauchy seq within a set not converging to an element of the set.

Let $S =$ positive reals. $X_n = \frac{1}{n}$. Cauchy, but has no limit in the set

12)

X is accumulation point of $\{X_n\}$ if every tail of the sequence gets arbitrarily close to X.

~~Another~~ Only the tail matters. Could remove any finite # of items

Examples: $\{1, 0, 1, 0, 1, 1, 1, 1, 1, 1, \dots\}$

has one accumulation point $(x=1)$.

$\{1, 0, 1, 0, 1, 0, 1, 0, \dots\}$ has two accumulation points.

Let $\{X_n\}$ be an ordering of $\mathbb{Q}$. Any real number is an accumulation point ~~about~~ of X by the desity of rationals in the reals.

If X is an accumulation point of ~~an~~ $\{X_n\}$ there is a subsequence $\{X_{n_k}\}$ converging to X.

Construction: Let $n_1 = 1$

Let n_k be such that $|X_{n_k} - X| < \frac{1}{k}$

and $n_k \geq n_{k-1}$

13) Bolzano - Weierstrass Theorem

Every bounded seq in $\mathbb{R}$ has a convergent subsequence.

Example: ~~$\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{\ }$~~

$\{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, ...\}$

tries to put each item as far from existing items

Qualitatively, if you ~~must~~ put only many things in a finite region, they must cluster.

Nonexample: if drop boundedness assumption

$x_n = n$. has no ~~cluster~~ convergent subsequence

Proof: Dyadic partition