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Analysis I
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## Day 21 — Summary — Definition of integral by limits of step functions

127. A step function from $[a, b] \to E$, where $E$ is a normed vector space, is a function of the form

$$f(x) = w_i \text{ for } a_{i-1} < t < a_i,$$

where $a = a_0 \leq a_1 \leq \ldots \leq a_n = b$ is a partition of $[a, b]$. Denote the set of step functions as St$([a, b], E)$.

128. The integral of a step function on $[a, b]$ is defined as $I(f) = \sum_{i=1}^{n}(a_i - a_{i-1})w_i$.

129. St$([a, b], E)$ is a subspace of the space of all bounded maps from $[a, b]$ into $E$. The operator $I$ is a linear operator from this subspace to $E$ with bound $b - a$. That is, $\|I(f)\|_E \leq (b - a)\|f\|_\infty$.

130. The integral operator $I$ can be extended to the closure of St$([a, b], E)$. We will call this closure the space of regulated maps, Reg$([a, b], E)$.

131. The closure of St$([a, b], E)$ contains $C^0([a, b], E)$. It also contains the class of piecewise continuous functions.