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Analysis I
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Day 1— Summary — Real Numbers

1. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the natural numbers, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ be the integers.
2. Let $\mathbb{Q}$ be the rationals. If $x \in \mathbb{Q}$, then $x = n/m$, for $n, m \in \mathbb{Z}$ and $m \neq 0$. There are a countable number of rationals.
3. Let $\mathbb{R}$ be the reals. There are an uncountable number of reals. Each real number has a decimal representation (possibly two)
4. Some axioms of real numbers:
   (a) $(x + y) + z = x + (y + z)\ \forall x, y, z \in \mathbb{R}$ (additive associativity)
   (b) $0 + x = x + 0\ \forall x \in \mathbb{R}$ (additive identity)
   (c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $x + y = 0$ (additive inverse)
   (d) $\forall x, y \in R, x + y = y + x$ (additive commutativity)
   (e) $(xy)z = x(yz)\ \forall x, y, z \in \mathbb{R}$ (multiplicative associativity)
   (f) $1x = x\ \forall x \in \mathbb{R}$ (multiplicative identity)
   (g) $\forall x \neq 0, \exists y$ such that $yx = 1$ (multiplicative inverse)
   (h) $xy = yx\ \forall x, y \in \mathbb{R}$ (multiplicative commutativity)
   (i) $x(y + z) = xy + xz\ \forall x, y, z \in \mathbb{R}$ (distributivity)
5. Completeness axiom of reals:
   (a) Every non-empty set of reals which is bounded from above has a least upper bound. We denote the least upper bound of a set S by $\sup(S)$, which stands for the supremum of S. If S is unbounded from above, then we say that $\sup(S) = \infty$.
   (b) Similarly, every non-empty set S which is bounded from below has a greatest lower bound, $\inf(S)$, which stands for the infimum of S. If S is unbounded from below, then we say that $\inf(S) = -\infty$.
6. Properties of the reals
   (a) Triangle inequality: For real numbers, $|x + y| \leq |x| + |y|$ and $|x - y| \geq |x| - |y|$.
   (b) Archimedian property: If $0 \leq x \leq 1/n\ \forall n \in \mathbb{N}$, then $x = 0$
   (c) Density of rationals within the reals: For all $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $q \in \mathbb{Q}$ such that $|q - x| < \varepsilon$.
   (d) Between two distinct rationals, there is a real. Between two distinct reals, there is a rational.
7. The sequence $\{x_n\}_{n=1}^{\infty}$ converges if $\exists\, a \in \mathbb{R}$ such that for all $\varepsilon > 0\ \exists N$ such that $n \geq N \Rightarrow |x_n - a| < \varepsilon$. We say that $\lim_{n \to \infty} x_n = a$.
8. A bounded monotonic sequence converges.

Analysis: Motivation

Example: ~~Sigas~~ Time Synchronization

~~Many problems are of form~~ ~~$\min f(x_1, \dots, x_n)$~~

Eg Say we have noisy measurements $X_i - X_j \approx b_{ij}$

Then $\displaystyle \min_{X_1 \cdots X_n} \underbrace{\sum_{i,j} \tfrac{1}{2}(X_i - X_j - b_{ij})^2}_{f(X_1 \cdots X_n)}$ will find best guess for $X_1 \cdots X_n$

To Solve: consider gradient descent

$$X^{(0)} = 0 \qquad X^{(i+1)} = X^{(i)} - \alpha \nabla f(X^{(i)})$$

Does $X^{(i)}$ converge to the true minimizer?

Example: Navier Stokes

$$\rho\left(\frac{dV}{dt} + V \cdot \nabla V\right) = -\nabla P + f \qquad \text{(Now viscosity)}$$

Approximate with a tiny viscosity

$$\rho\left(\frac{dV^{(\varepsilon)}}{dt} + V^{(\varepsilon)} \cdot \nabla V^{(\varepsilon)}\right) = -\nabla P^{(\varepsilon)} + \varepsilon \Delta V^{(\varepsilon)} + f$$

Do solutions exist?

As $\varepsilon \to 0$, does $V^{(\varepsilon)}$ converge? If so, does it satisfy N-S?

Key ideas:

Convergence (of reals, functions)

Compactness (way of getting convergence)

Interchanging limits and derivatives/integrals

Proposition: There is no $X \in \mathbb{Q}$ s.t. $X^2 = 2$.

Proof: Let $X = m/n$. WLOG m is odd or n is odd.

If $X^2 = 2$, then $\frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$

$\Rightarrow$ m is Even. So $m = 2k$ for some k $\Rightarrow (2k)^2 = 2n^2$

$\Rightarrow 2k^2 = n^2 \Rightarrow$ n Even $\times$.

5 A least upper bound for $S \subset \mathbb{R}$ is the smallest b such that $x \leq b \;\; \forall x \in S$.

That is, $x \leq b \;\; \forall x \in S$ AND $x \leq a \; \forall x \in S \Rightarrow b \leq a$.

~~Does a l.u.b. exist for subsets of $\mathbb{Z}$, $\mathbb{N}$~~

Proposition: Every non empty ~~bounded~~ $S \subset \mathbb{R}$ bounded from above has a l.u.b.

Example: $S = \{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \}$

lub $= 1$

Check: $\forall x \in S$, $x \leq 1$.

If $a < 1$, $\exists \, x > a$ by Archimedean property.

Nonexample: if S is unbounded: $\mathbb{Z}$ has no l.u.b.

Why ~~is~~ only one sided boundedness?

Example: Negative integers has l.u.b. of -1.

Does this property hold for subsets of ~~$\mathbb{R}$~~, $\mathbb{N}$, $\mathbb{Z}$ that are bounded from above? Yes. ~~all subsets of $\mathbb{R}$, but~~

Then whats different between subsets of $\mathbb{Z}$ that are bdd from above and subsets of $\mathbb{R}$ bdd from above?

bdd subsets of $\mathbb{Z}$ achieve max — similar to (well ordering principle)

bdd subsets of $\mathbb{R}$ may or may not

Does least upperbound property hold for $\mathbb{Q}$?

No. Take $\{3, 3.1, 3.14, 3.141, 3.1415, \ldots\}$

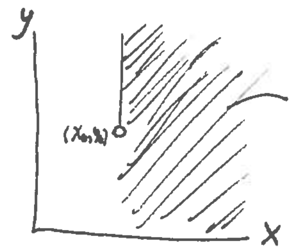
l.u.b (in $\mathbb{R}$) is $\pi \notin \mathbb{Q}$

6d) Neat application:

There is no $U(x,y)$ such that $U(x_1,y_1) > U(x_2,y_2)$

when $x_1 > x_2$ or $(x_1 = x_2$ & $y_1 > y_2)$

Graphically, want a $U(x,y)$ st



$U(x,y) > U(x_0,y_0)$ in this region.

Suppose such a U exists

Proof: For each $x \in \mathbb{R}$, ~~there is a rat~~

$U(x,1) > U(x,0)$ and there is a rational inbetween.

That is, $\forall x \in \mathbb{R}\ \exists\ f(x) \in \mathbb{Q}$

Note $f(x_1) > f(x_2)$ if $x_1 > x_2$.

So for all $\mathbb{R}$, we have identified a unique corresponding elt in $\mathbb{Q}$.

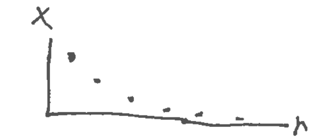
Impossible by cardinality $\times$.

depends on ε

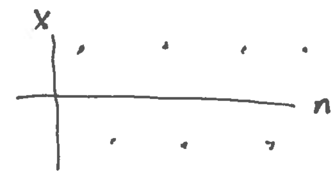
7(m)) $x_n \to a$ if $\forall \varepsilon\ \underbrace{\exists N \text{ st } n \geq N}\ \Rightarrow |x_n - a| < \varepsilon$

x_n ~~gets arbitrarily close to~~ is within ε of a for sufficiently large n.

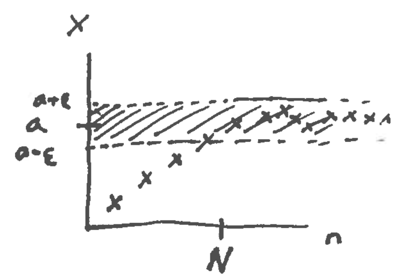
Example: $x_n = \frac{1}{n}$ $\quad x_n \to 0$



Non Example: $x_n = (-1)^n$



Visually: A seq x_n conv. to a if for any strip about a, the sequence is eventually contained in the strip.



8) A bounded monotonic sequence converges

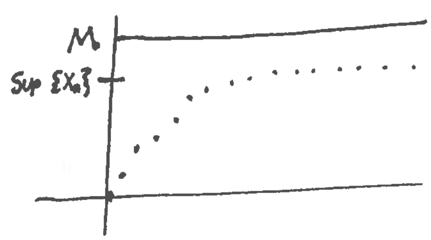
We say $\{X_k\}_{k\geq 1}^{\infty}$ is bounded if $\exists$ M st $|X_k| \leq M \quad \forall k$

We say $\{X_k\}_{k\geq 1}^{\infty}$ is monotone increasing if $X_{k+1} \geq X_k \quad \forall k$
______ ______ decreasing if $X_{k+1} \leq X_k \quad \forall k$

Example: Let $X_n = 1 - \frac{1}{n}$. $\{X_n\}$ is monotonic increasing, bounded by 1, converges to 1.

Nonexamples: $\{n\}$ is increasing, not bounded, does not converge
$\{(-1)^n\}$ is bounded, not monotonic, does not converge

Visually:



Roughly any seq that stays below M and is increasing must level off (to its sup)

Claim: If $\{X_n\}$ is monotone increasing, bounded, it converges to $\sup_n \{X_n\}$

Pf: (Gist) Let $a = \sup \{X_n\}$
Fix any ε, $a - \varepsilon$ is not l.u.b. of $\{X_n\} \Rightarrow \exists N$ st $X_N \geq a - \varepsilon$
By monotonicity $a - \varepsilon \leq X_n \quad \forall n \geq N$
By ~~bddness~~ dfn of sup, $X_n \leq a \quad \forall n$

So $$|X_n - a| \leq \varepsilon \quad \forall n \geq N.$$

Application: Optimization. If $f(X_n)$ is decreasing and f bounded from below, then $f(X_n)$ converges.