27 October 2015
Analysis I
Paul E. Hand
hand@rice.edu

Day 16 — Summary — Equivalence relations

94. A relation, $\sim$, on a set X is an equivalence relation if it is reflexive, symmetric, and transitive. That is, if for all $a, b, c \in X$

    (a) $a \sim a$ (reflexivity)

    (b) $a \sim b \Rightarrow b \sim a$ (symmetry)

    (c) $a \sim b$ and $b \sim c \Rightarrow a \sim c$ (transitivity)

95. Given a set X and an equivalence relation $\sim$, the equivalence class of an element $a \in X$ is the set of elements equivalent to a. The set of equivalence classes is denoted by $X \,/ \sim$. We can define operations (e.g. addition, multiplication) on equivalence classes if the operation is well defined (is independent of which representative is chosen from the equivalence classes).

96. We can define an equivalence relation between two Cauchy sequences of a (not necessarily complete) normed vector space:

$$\{x_n\} \sim \{y_n\} \text{ if and only if } \lim_{n\to\infty} (x_n - y_n) = 0$$

    The set of equivalence classes forms a normed vector space.

Equivalence is a measure of "Sameness"

94 a) Example of equivalence relation on $\mathbb{Z}$

$$n \sim m \iff n = m + 2k \quad \text{for } k \in \mathbb{Z}$$

$$(n \equiv m \mod 2)$$

Proof it is equivalence relation

$n \sim m$ as $n = m + 2 \cdot 0$

$n \sim m \Rightarrow m \sim n$ as $n = m + 2k \Rightarrow m = n + 2(-k)$

$n \sim m$ & $m \sim p \Rightarrow n \sim p$ as $\begin{matrix} n = m + 2k \\ m = p + 2j \end{matrix} \Rightarrow n = p + 2(k + j)$

94 ~~2~~) Activity Equivalence Relation

$x \in \mathbb{Z}$, $y \in \mathbb{Z}$ $x \sim y$ iff tens digit of x = tens digit of y (in decimal)

$x \in \mathbb{R}$, $y \in \mathbb{R}$ $x \sim y$ iff tens digit of x = tens digit of y (in decimal)

$x \in \mathbb{N}$ $y \in \mathbb{N}$ $x \sim y$ if x & y have common prime factor

$f \in C(\mathbb{R})$ $g \in C(\mathbb{R})$ $f \sim g$ if $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ exists and equals 1.

Example:

95 ~~2~~) If $\sim$ is relation on $\mathbb{Z}$ $x \sim y$ iff $x \equiv y \pmod 2$

The equivalence class of 0 is set of evens

— — — 1 is set of odds

$\mathbb{Z}/\sim$ = { evens, odds }

= $\{[0], [1]\}$

Example: $x \in \mathbb{Z}, y \in \mathbb{Z}$ $x \sim y$ if 10's digit of x = 10's digit of y (in decimal)

What are equivalence classes?

$\mathbb{Z}/\sim$ = { #'s w/ 0 in tens digit, #'s w/ 1 in 10's dig, ... #'s w/ 9 in 10's digit }

$\{[0], [10], [20], [30], \ldots, [90]\}$

identify each equiv class with an element

~~Addition is~~

Example:

Addition is well defined on Equivalence class of integers mod p

Let $x, y \in \mathbb{Z}$. Let $x \sim y \Rightarrow x = y + kp$ for $k \in \mathbb{Z}$.

To show $[x] + [y]$ is well defined, chose arbitrary els of $[x]$ & $[y]$

$$x + k_1 p + y + k_2 p = x + y + (k_1 + k_2) p \in [x+y]$$

96 ~~B~~)

Let $\{x_n\}_{n=1}^{\infty}$ be Cauchy in a normed vector space V

Let $\{y_n\}_{n=1}^{\infty}$ — — — — — — V

Let $\{x_n\} \sim \{y_n\} \iff \lim_{n\to\infty} x_n - y_n = 0 \iff \forall \varepsilon \, \exists N \text{ st } n \geq N \Rightarrow |x_n - y_n| < \varepsilon$

Equivalence Classes are a Vector space

$$[\{x_n\}] + [\{y_n\}] = [\{x_n + y_n\}]$$

Pf: $\forall \varepsilon \, \exists N_1 \text{ st } m,n \geq N_1 \Rightarrow |\hat{x}_n - \hat{x}_m| < \varepsilon/2$

Let $\{\hat{x}_n\}$ be Cauchy and $\lim_{n\to\infty} x_n - \hat{x}_n = 0$

Let $\{\hat{y}_n\}$ be — — $\lim_{n\to\infty} y_n - \hat{y}_n = 0$ $\quad \forall \varepsilon \, \exists N_2 \text{ st } m,n \geq N_2 \Rightarrow |\hat{x}_n - \hat{y}_n| < \varepsilon/2$

$\{\hat{x}_n + \hat{y}_n\}$ is Cauchy. Fix ε Let $N = N_1 \vee N_2$. $\forall m,n \geq N$, $|\hat{x}_n + \hat{y}_n - \hat{x}_m - \hat{y}_m|$

$$\leq |\hat{x}_n - \hat{x}_m| + |\hat{y}_n - \hat{y}_m|$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Now, show $\{\hat{x}_n + \hat{y}_n\} \in [\{x_n + y_n\}]$

$$\lim_{n\to\infty} \hat{x}_n + \hat{y}_n - x_n - y_n = \lim_{n\to\infty} (\hat{x}_n - x_n) + (\hat{y}_n - y_n) = \lim_{n\to\infty} \hat{x}_n - x_n + \lim_{n\to\infty} \hat{y}_n - y_n$$

$$= 0 + 0 = 0. \quad \blacksquare$$

$$[c\{x_n\}] = [\{cx_n\}] = c[\{x_n\}]$$

Equivalence classes or Cauchy seq have a norm.

$$\|[\{x_n\}]\| = \lim_{n\to\infty} \|x_n\|$$

Pf: If $\{x_n\}$ Cauchy then $\|x_n\|$ Cauchy. Has a limit L.

If $\{\hat{x}_n\}$ Cauchy & $\|x_n - \hat{x}_n\| \to 0$ as $n\to\infty$ then $\lim \|\hat{x}_n\| = \lim \|x_n\| = L$

~~Let~~ $\cancel{\lim_{n\to\infty} \|\hat{x}_n\| = \lim \|\hat{x}_n\|}$. Let $L = \lim \|x_n\|$

$\forall \varepsilon \, \exists N_1 \text{ st } n \geq N_1 \Rightarrow \|x_n - \hat{x}_n\| < \varepsilon$

$\forall \varepsilon \, \exists N_2 \text{ st } n \geq N_2 \Rightarrow \big| \|x_n\| - L \big| < \varepsilon$

~~If $n \geq \max(N_1, N_2)$~~, $\big| \|\hat{x}_n\| - L \big| = \big| \|x_n + \hat{x}_n - x_n\| - L \big|$

Suffices to show $\lim_{n\to\infty} \|x_n\| - \|\hat{x}_n\| = 0$. Fix ε. $\exists N \text{ st } \|x_n - \hat{x}_n\| < \varepsilon \; \forall n \geq N$

Hence $\|x_n\| - \|\hat{x}_n\| < \varepsilon$ & $\|\hat{x}_n\| - \|x_n\| < \varepsilon$, so $\big| \|x_n\| - \|\hat{x}_n\| \big| < \varepsilon$, $\forall n \geq N$ $\blacksquare$