8 October 2015
Analysis I
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Day 12 — Summary — Complete Normed Vector Spaces

63. Definition: A sequence x_n in a normed vector space is Cauchy if

$$\forall \varepsilon \; \exists N \text{ such that } n, m \geq N \Rightarrow \|x_n - x_m\| < \varepsilon.$$

64. In a normed vector space, we say that x_n converges to x if $\forall \varepsilon \; \exists N$ such that $n \geq N \Rightarrow \|x_n - x\| < \varepsilon$. We write this as $\lim_{n\to\infty} x_n = x$

65. Definition: A vector space is complete if any Cauchy sequence converges to an element in the set.

66. Definition: A Banach space is a complete normed vector space.

67. Definition: $\mathbb{R}^n$ is a Banach space under the ℓ_∞ norm. By equivalence of norms on finite dimensional spaces, it is a Banach space under any norm.

Warmup:

Fix an N.

b) Find a sequence X such that $\|X\|_2 = 1$ and $\|X\|_1 = N$

a) Fix an N

Find a seq X s.t. $\|X\|_\infty = 1$ and $\|X\|_1 = N$

67) Example: Space that isn't complete wrt a norm

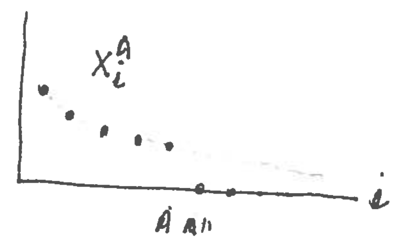
l_1 not complete wrt l_∞ norm

$$V = l_1 = \{ X \text{ sequence} \mid \sum_{i=1}^{\infty} |X_i| < \infty \}$$

Pf: Suffices to exhibit ~~Cauch~~ seq $X^i \in V$ that is Cauchy wrt l_∞ but does not converge to anything in V.

$$\text{Let } X^n_i = \begin{cases} \frac{1}{i} & \text{if } i \le n \\ 0 & \text{o'wise} \end{cases}$$

$$= 1_{[1,n]}(i) \cdot \frac{1}{i}$$



Note: $\|X^n - X^m\|_\infty = \frac{1}{n \wedge m}$

This seq is Cauchy. Fix ε. Let $N > \frac{1}{\varepsilon}$.

$\forall\, n, m \geq N \quad \|X^n - X^m\|_\infty \leq \frac{1}{N} < \varepsilon$.

There is no $X^\infty \in l_1$ such that $X^n \to X^\infty$ in l_∞.

Informally: B/c $X^n \to \{\frac{1}{i}\}_{i=1}^{\infty}$ which is not in l_1

Formally: Each $X^n \in l_\infty$. l_∞ is complete wrt l_∞ norm. $X^n \to \{\frac{1}{i}\}_{i=1}^{\infty}$ in l_∞ norm.
Limits are unique, so if $X^n \to X^\infty \in l_1 \subset l_\infty$, $X^\infty = \{\frac{1}{i}\}_{i=1}^{\infty} \notin l_1$. ∎

Exercise: l_∞ is complete under the l_∞ norm.

6.6 ~~4~~ 8) $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete.

Proof:

Consider X^i a Cauchy seq in $\mathbb{R}^n$ under $\|\cdot\|_\infty$

$\forall \varepsilon \; \exists N$ st $n,m \geq N \implies \|X^n - X^m\|_\infty < \varepsilon$

As $|X_j^n - x_j^m| \leq \|X^n - x^m\|_\infty$ we hav

$\forall \varepsilon \; \exists N$ st $n,m \geq N \implies \| X_j^n - x_j^m|_\infty < \infty$.

So each component is Cauchy and has limit $X_j^\infty \in \mathbb{R}$.

~~Then~~
Remains to show $X^i \to X^\infty$ under $\|\cdot\|_\infty$.

$\forall \varepsilon \; \exists N_j$ st $|X_j^n - x_j^\infty| < \varepsilon \quad \forall \; n \geq N$

Fix ε. Let $N = \max N_1 \dots N_n$. Then $\|X^n - x^\infty\| < \varepsilon$ ∎