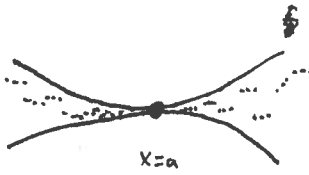
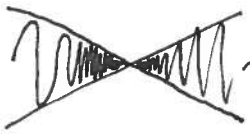
1) Squeeze Theorem

Idea: If a function is sandwiched between two functions that share a limit, then all three share the limit.

Picture:



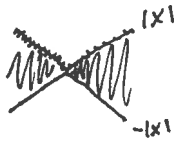
— this function is not continuous, but it has a limit as $x \to a$



— this function gets arbitrarily steep, but still has a limit

Application: $g(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin\frac{1}{x} & \text{if } x \neq 0 \end{cases}$ is continuous at $x = 0$.

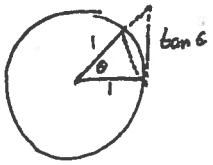
It is squeezed between $|x|$ and $-|x|$.

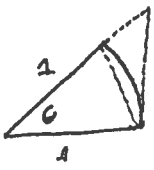


Hence it has limit 0

Application: Show $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Pf:





Area of small triangle $\leq$ area of sector $\leq$ area of big triangle

$$\frac{\sin\theta}{2} \leq \frac{\theta}{2} \leq \tan\theta$$

$$\Rightarrow \quad 1 \leq \frac{\theta}{\sin\theta} \leq \frac{1}{\cos\theta}$$

$$\Rightarrow \quad 1 \geq \frac{\sin\theta}{\theta} \geq \cos\theta.$$

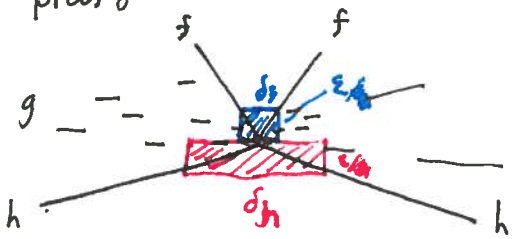
Squeezed $\frac{\sin\theta}{\theta}$ between 1 & $\cos\theta$ which $\to 1$ as $\theta \to 0$

Formal Statement:

If $f(x) \le g(x) \le h(x)$ for x sufficiently near a

and if $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$

Gist of proof:



For a given ε, choose the smaller of the δ_f & δ_h corresponding to $\frac{\varepsilon}{2}$ error from the limit. Because g squeezes between, its error is $\le \varepsilon$ within δ_f & δ_h.