18 September 2014
Analysis I
Paul E. Hand
hand@rice.edu

HW 4 [Revised 18 Sep]

Due: Sep 23 in class. Justify all of your work.

1. (Revised) Directly prove that $f(x) = x^2$ is Riemann integrable on $[0, 1]$ and that the value of the Riemann integral is $1/3$. Do this by showing that the supremum of all lower sums equals the infimum of all upper sums equals $1/3$.

2. (Revised) Evaluate
$$\lim_{n\to\infty} \frac{\log n + \log(n+1) + \cdots + \log(2n-1)}{n} - \log n$$
by viewing it as a Riemann sum. Feel free to use elementary techniques of integration to evaluate the value of the resulting integral. Make sure to justify why the limit exists.

3. Let $f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$
Is f Riemann integrable on $[0, 1]$? Prove it. If so, what is the value of the integral.

4. Let $f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \sin(1/x) & \text{otherwise.} \end{cases}$
Is f Riemann integrable on $[0, 1]$? Prove it.

5. (Revised) Suppose that the sequence of functions $f_n(x)$ converges uniformly to $f(x)$ on $[a, b]$. That is, suppose that $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_n(x) - f(x)| = 0$.
   (a) Prove that for all x, $\lim_{n\to\infty} f_n(x) = f(x)$
   (b) Prove that f is Riemann integrable and that
   $$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$
   Feel free to use facts like $|\int_a^b g(x)dx| \leq \int_a^b |g(x)|dx$.

6. If you were to present Theorem VI.2.1 (Cauchy Schwarz Inequality) in class, write up the notes of what you would say.