Brief Notes on Dimension Theory

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Throughout, V is a vector space, as defined in VI.1 of Lang, where you will also find definitions of *linear combination*, the vector space $\mathbf{R}^n$, and the vector space $\mathcal{F}(S, V)$ of functions on a set S with values in a vector space V (Example 8, with different notation). Recall that I gave a careful recursive definition of linear combination in class.

A linear combination is *nontrivial* if at least one coefficient is nonzero.

Defn. Suppose $S \subset V$. Then Span(S) denotes the set of all linear combinations of elements of S:

$$\mathrm{Span}(S) = \{c_1v_1 + ... + c_kv_k : k \in \mathbf{Z}^+, c_1, ..., c_k \in \mathbf{R}, v_1, ...v_k \in S\}$$

Defn. A set $S \subset V$ is *linearly dependent* if there exists some nontrivial linear combination of elements of S equal to the zero vector, i.e. for some $k \in \mathbf{Z}^+, c_1, ..., c_k \in \mathbf{R}, v_1, ...v_k \in S$, with $|c_1| + |c_2| + ... + |c_k| > 0$,

$$c_1v_1 + ... + c_kv_k = 0.$$

S is *linearly independent* if it is not linearly dependent.

Examples:

1. the subset $\{0\} \subset V$ (consisting of the zero vector) is always linearly dependent. So is any set containing the zero vector.

2. in the vector space $\mathbf{R} = \mathbf{R}^1$, the set $\{1\}$ is linearly independent. However the set $\{1, 2\}$ is linearly dependent, since $2 \cdot 1 + (-1) \cdot 2 = 0$.

3. in the vector space $\mathbf{R}^2$, set $v_1 = (1,0), v_2 = (1,-1)$. Then $\{v_1, v_2\}$ is linearly independent: if $c_1v_1 + c_2v_2 = (c_1 + c_2, -c_2) = (0,0)$, then $c_1 = c_2 = 0$.

4. in the vector space $\mathbf{R}^n$, $n \in \mathbf{Z}^+$, for each $k \in J_n$ set e_k = vector with kth coordinate = 1, all other coordinates = 0. e_k is called the *kth standard basis vector*. Any subset of $S = \{e_1, ..., e_n\}$ is linearly independent.

Lemma 1. Suppose $T \subset S \subset V$, and T is linearly dependent. Then S is linearly dependent.
Proof: The hypothesis supposes the existence of $k \in Z^+$, $t_1, ..., t_k \in T, c_1, .., c_k \in \mathbf{R}$ so that

$$c_1t_1 + ... + c_kt_k = 0, \; c_j \neq 0 \, \text{for some} \, j \in J_k.$$

However $t_1, ..., t_k \in S$ also, so S is linearly dependent. **Q.E.D.**

Lemma 2. Suppose $T \subset S \subset V$, and S is linearly independent. Then T is linearly independent.
Proof: Else, by Lemma 1, S would be linearly dependent. **Q.E.D.**

Lemma 3. Suppose $T \subset V$ is linearly independent, and $t \notin \text{Span}(T)$. Then $T \cup \{t\}$ is linearly independent.
Proof: Suppose not: that is, there exist $t_1, ..., t_k \in T$, $c, c_1, ..., c_k \in \mathbf{R}$ so that

$$ct + c_1t_1 + ... + c_kt_k = 0. \tag{1}$$

Either $c \neq 0$ or $c = 0$. In the former case, divide the preceding equation through by c and rearrange to get

$$t = \left(-\frac{c_1}{c}\right) t_1 + ... + \left(-\frac{c_k}{c}\right) t_k \;\Rightarrow\; t \in \text{Span}(T),$$

contradicting the second assumption. In the latter case, the linear combination (1) must be a nontrivial linear combination of elements of T, which contradicts the first assumption. **Q.E.D.**

Defn. A finite subset $S \subset V$ is a *basis* of V if and only if (1) S is linearly independent, and (2) $V = \text{Span}(S)$.

Theorem 1. Suppose that $S \subset V$ is a basis, and $\#S = n \in \mathbf{Z}^+$. Suppose $T \subset V$, and either T is infinite or $\#T > n$. Then T is linearly dependent.

Proof: Suppose not, that is, that T is linearly independent.

Claim: for each $k = 0, ...n$, there exists $S_{n-k} \subset S$, $T_k \subset T$ so that $\#S_{n-k} = n-k$, $\#T_k = k$, and $S_{n-k} \cup T_k$ is a basis of V. Establish the claim by induction: for $k = 0$, $S = S_n$, $T_0 = \emptyset$, and the claim is just the hypothesis that S is a basis. Suppose the claim to be true for $k < n$. Enumerate the members: $S_{n-k} = \{v_1, .., v_{n-k}\}, T_k = \{w_1, ..., w_k\}$. Since $\#T_k = k < n < \#T$, there exists at least one $t \in T \setminus T_k$. Since $S_{n-k} \cup T_k$ is a basis, can choose $c_1, .., c_{n-k}, d_1, ..., d_k \in \mathbf{R}$ so that

$$t = c_1 v_1 + ... + c_{n-k} v_{n-k} + d_1 w_1 + ... + d_k w_k.$$

Note that at least one of the c_j must be nonzero, else the preceding equation would show that T is linearly dependent. Renumber the $v's$ (i.e. compose the enumeration map $J_{n-k} \rightarrow S_{n-k}$ with a permutation of J_{n-k} so that $j = n-k$. Then you can solve the above equation for v_{n-k}:

$$v_{n-k} = \left(-\frac{c_1}{c_{n-k}}\right) v_1 + ... \left(-\frac{c_{n-k-1}}{c_{n-k}}\right) v_{n-k-1}$$

$$+ \left(\frac{-1}{c_{n-k}}\right) t + \left(-\frac{d_1}{c_{n-k}}\right) w_1 + ... \left(-\frac{d_k}{c_{n-k}}\right) w_k.$$

Rename $w_{k+1} = t$, set $S_{n-k-1} = \{v_1, ..., v_{n-k-1}\}, T_{k+1} = \{w_1, ..., w_{k+1}\}$. These sets have the right cardinalities, so it remains only to show that $S_{n-k-1} \cup T_{k+1}$ is a basis. To see that this set is linearly independent, suppose that for some $a_1, ..., a_{n-k}, b_1, ..., b_k \in \mathbf{R}$,

$$0 = a_1 v_1 + ... + a_{n-k-1} v_{n-k-1} + b_1 w_1 + ... + b_k w_k + b_{k+1} w_{k+1}$$

and substitute the expression given above for $w_{k+1} = t$:

$$= a_1 v_1 + ... + a_{n-k-1} v_{n-k-1} + b_1 w_1 + ... + b_k w_k +$$

$$b_{k+1}(c_1 v_1 + ... + c_{n-k} v_{n-k} + d_1 w_1 + ... + d_k w_k)$$

$$= (a_1 + b_{k+1} c_1) v_1 + ... + (a_{n-k-1} + b_{k+1} c_{n-k-1}) v_{n-k-1}$$

$$+ b_{k+1} c_{n-k} v_{n-k}$$

$$+ (b_1 + b_{k+1} d_1) w_1 + ... + (b_k + b_{k+1} d_k) w_k.$$

Since $S_{n-k} \cup T_k$ is a basis, all coefficients in this linear combination must vanish. In particular, $b_{k+1} c_{n-k} = 0$. However $c_{n-k} \neq 0$, so $b_{k+1} = 0$, and therefore $a_1 = ... = a_{n-k-1} = b_1 = ... = b_k = 0$ also, that is, the linear combination is trivial.

It's equally easy to see that $S_{n-k-1} \cup T_{k+1}$ spans V, thus finishing the induction step and therefore the proof of the claim.

In particular, for $k = n$ we have shown the existence of a *basis* $T_n \subset T$ with $\#T_n = n$ (T_n is a basis all by itself, since $S_0 = \emptyset$). But $\#T > n$, so there is $t \in T \setminus T_n$. Since $t \in \text{Span}(T)$, the set $T_n \cup \{t\} \subset T$ must be linearly dependent, but then so must be T, a contradiction. **Q.E.D.**

Theorem 2 (Main Theorem of Dimension Theory): For any vector space V, either

- V has no basis, or
- all bases of V have the same cardinality.

Defn. If V has a basis, then the *dimension* of V, written dim$V$, is the cardinality of any basis. That is, if S is (any) basis of V, then

$$\text{dim}V = \#S.$$

If V has a basis, it is called *finite-dimensional*. By convention, the trivial vector space $V = \{0\}$, which clearly has no basis, has dimension zero. If V does not have a basis and contains a nonzero vector, V is called *infinite-dimensional*.

Theorem 3. Suppose that dim$V = n \in \mathbf{Z}^+$, and $S \subset V$ spans V (that is, $V = \text{Span}(S)$). Then there exists a basis B of V with $B \subset S$.
Proof: Let

$$K = \{k \in \mathbf{Z}^+ : \text{ there exists a linearly independent subset } T \subset S \text{ with } \#T = k\}.$$

It follows from Theorem 1 that $K \subset J_n$, so K has a maximum member, say m, hence there is a subset $\{v_1, ..., v_m\} \subset S$ which is linearly independent. Suppose $\{v_1, ..., v_m\}$ does *not* span V, then there must exist a $w \in S \setminus \text{Span}(\{v_1, ..., v_m\})$ - otherwise, $S \subset \text{Span}(\{v_1, ..., v_m\})$, whence $V = \text{Span}(S) \subset \text{Span}(\{v_1, ..., v_m\})$. Then (by Lemma 3) $\{w, v_1, ..., v_m\}$ is a linearly independent subset of S, which contradicts the maximality of m. Conclude that in fact $\{v_1, ..., v_m\}$ spans V, so is a basis (and $m = n$). **Q.E.D.**