23 September 2014
Analysis I
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Day 9 — Summary — Taylor Series and Normed Vector Spaces

1. Big oh and Little oh notation:
   (a) $f(x) = o(g(x))$ as $x \to x_0$ means that $f(x)/g(x) \to 0$ as $x \to x_0$
   (b) $f(x) = O(g(x))$ as $x \to x_0$ means that there exists C such that $|f(x)| \leq Cg(x)$
2. A Taylor series is a local approximation of a function, and it is obtained by matching the value and a given number of derivatives of that function at a particular point.
3. The nth order Taylor series of $f(x)$ about $x = a$ is given by
$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n$$
4. The nth Taylor remainder term is
$$R_n(x) = f(x) - \left(f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}}{n!}(x-a)^n \right).$$
5. The nth order Taylor series is accurate to the $n+1$st order in the neighborhood of the point of expansion. The constant factor of the error term is controlled by the maximum value of the $n+1$st derivative of the function.

   If $f \in C^{n+1}$ in a neighborhood of a, then $R_n(x) = O(|x-a|^{n+1})$ as $x \to a$. More precisely,
$$R_n(x) \leq \max |f^{(n+1)}| \cdot \frac{|x-a|^n}{n!}.$$
   The max is taken over the neighborhood and the inequality holds for all points in the neighborhood.
6. A vector space V over the reals is a set that permits addition and scalar multiplication.
   (a) $(x+y)+z = x+(y+z)\ \forall x, y, z \in V$
   (b) $0 + x = x\ \forall x \in V$
   (c) $\forall x \in V, \exists y \in V$ such that $x + y = 0$
   (d) $x + y = y + x\ \forall x, y \in V$
   (e) For $x \in V$ and $a, b \in \mathbb{R}$, $(ab)x = a(bx), (a+b)x = ax + bx, a(x+y) = ax + ay$.
7. A norm on a vector space V is denoted by $\|\cdot\|$ and satisfies
   (a) $\|x\| \geq 0$ for all $x \in V$
   (b) $\|x\| = 0 \Leftrightarrow x = 0$.
   (c) $\|ax\| = |a|\|x\|$ for all $x \in V, a \in \mathbb{R}$
   (d) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in V$

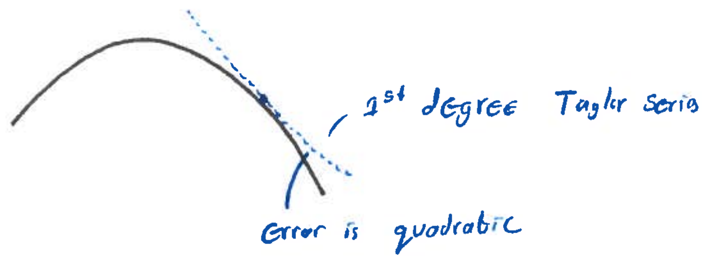
Warmup

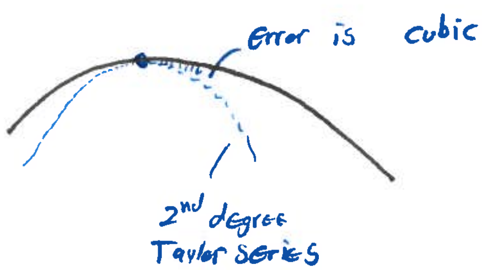
Find $f(x)$ st $f(x) = O(\frac{1}{x^n})$ as $x \to \infty$ $\forall n$.

Find $g(x)$ st $g(x) = O(x^n)$ as $x \to 0$ $\forall n$.

Taylor Remainder Theorem

An n^{th} order Taylor series has local error on $n+1^{st}$ order





Precise statement:

Let $f \in C^n$ in a nbh of $x=0$.

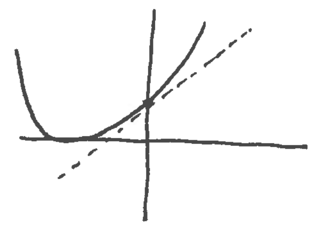
Let $R_n(x) = f(x) - \left[f(0) + \cdots + \frac{f^{(n-1)}}{n-1!}(x-0)^{n-1} \right]$

Then $|R_n(x)| \leq \underbrace{\left(\max_t f^{(n)}(t) \right)}_{\text{max of } n^{th} \text{ deriv sets the constant}} \cdot \frac{|x|^n}{n!}$

$|R_n(x)| \leq O(|x|^n)$

Example: Of a function in C^2 such that error term from 1st order Taylor series is optimal

$f(x) = (x+1)^2$ at $x = 0$



Taylor series about $x=0$

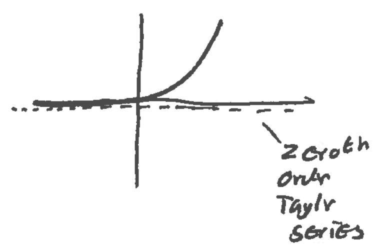
$f(x) \approx 1 + 2x$

~~Thm~~ Theorem guarantees $|f(x) - (1+2x)| \le 2\,\frac{|x|^2}{2} = |x|^2$ for small x

Actual error: $(x+1)^2 - (1+2x) = x^2+2x+1-1-2x = x^2$ ✓

Example: Search for a $f \in C^1$, $f \notin C^2$ such that error term of 0^{th} order Taylor series is optimal.

Consider $f(x) = \begin{cases} 0 & x \le 0 \\ x^2 & x \ge 0 \end{cases}$



Taylor series about $x=0$

$f(x) \approx 0 + 0x$

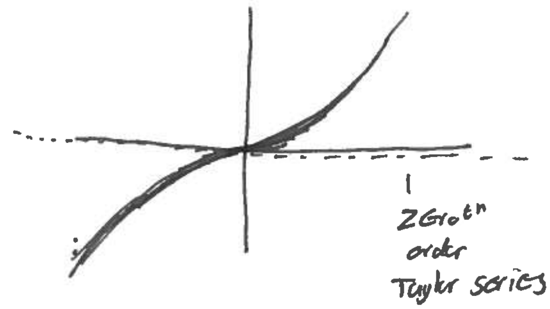
Thm guarantees $|f(x) - 0| \le 2|x|$

Actual error $|f(x) - 0| \le |x|^2$ which is better than theorem guarantees.

Example: Want $f \in C^1$, $\notin C^2$ s.t. Zeroth order Taylor series error is optimal

Let $f_n(x) = x^{1+\frac{1}{n}}$ for n odd, positive integer

Note $f_n \in C^1$
$f_n \notin C^2$



~~Theorem guarantees~~

~~First order~~
Zeroth order Taylor series

$f_n(x) \approx f_n(0) = 0$

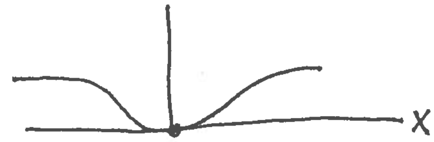
Theorem guarantees: $|f_n(x) - 0| \leq |x|$ on $[-1, 1]$

Actual error: $|f_n(x) - 0| = |x|^{1+\frac{1}{n}}$ which is a tiny bit better

As $n \to \infty$, we reach optimal error estimate
(note $n \to \infty$ limit is $f_\infty(x) = x$)

Example: If $f \in C^\infty$, is infinite Taylor series exact?
or is it merely more accurate than any power of x.

$$f(x) = \begin{cases} e^{-\frac{1}{|x|}} & x \neq 0 \\ 0 & x = 0 \end{cases} \in C^\infty(\mathbb{R})$$



Taylor series about $x=0$ of any order is $f(x) \approx 0$.

Theorem guarantees $|f(x) - 0| \leq C_n |x|^n \quad \forall n$.

Actual error is bdd by $e^{-\frac{1}{|x|}}$ which is faster decaying than any power of x as $x \to 0$

Example of a function nowhere equaling its Taylor series (except at single point)

Application: Order of accuracy of discretization of a derivative.

If f smooth, $\dfrac{f(\Delta x)-f(0)}{\Delta x} \approx f'(0) + O(\Delta x)$. first order

$$\frac{f(\Delta x)-f(-\Delta x)}{2\Delta x} \approx f'(0) + O(\Delta x^2) \quad \text{Second order}$$

First order:

Proof: $f(\Delta x) = f(0) + f'(0)\Delta x + R_1(\Delta x)$

so $\dfrac{f(\Delta x)-f(0)}{\Delta x} = f'(0) + \dfrac{R_1(x)}{\Delta x}$.

We know $|R_1(x)| \leq \max f'' \cdot \dfrac{\Delta x^2}{2}$

So $\dfrac{f(\Delta x)-f(0)}{\Delta x} = f'(0) + O(\Delta x)$

Second order:

$$f(\Delta x) = f(0) + f'(0)\Delta x + f''(0)\frac{\Delta x^2}{2} + R_2(\Delta x)$$
$$f(-\Delta x) = f(0) + f'(0)(-\Delta x) + f''(0)\frac{\Delta x^2}{2} + \tilde{R}_2(\Delta x)$$

$$\frac{f(\Delta x)-f(-\Delta x)}{2\Delta x} = f'(0) + \underbrace{\frac{R_2(\Delta x)-R_2(-\Delta x)}{2\Delta x}}_{\leq \frac{\Delta x^3 + \Delta x^3}{\Delta x} = \Delta x^2}$$

so discretization accurate to 2^{nd} order.

Proof:

Taylor Series Remainder

$$f(x) = f(0) + \int_0^x f'(t)\,dt.$$

$\downarrow$ IBP

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)\, f''(t)\,dt$$

$\downarrow$ IBP

$$f(x) = f(0) + f'(0)x + \dots + f^{(n-1)}(0)\frac{x^{n-1}}{n-1!} + \underbrace{\int_0^x \frac{(x-t)^{n-1}}{n-1!} f^{(n)}\,dt}_{\text{Remainder term.}}$$

~~Use intermediate Value Theorem~~

Bound

$$\leq \max f^{(n)} \int_0^x \frac{(x-t)^{n-1}}{n-1!}\,dt$$

$$= \max f^{(n)} \left(- \frac{(x-t)^n}{n!} \Big|_0^x \right)$$

$$= \max f^{(n)} \frac{x^n}{n!}.$$

Examples: Is it a vector-space or not?

$\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$

$\{(x,y) \mid x+y=0\} \subset \mathbb{R}^2$

$\{(x,y) \mid x+y=1\} \subset \mathbb{R}^2$

$\{(x,y) \mid x=0 \text{ or } y=0\}$

$C(0,1)$

$C[0,1]$

All Sequences in $\mathbb{R}$ $\{x_1, x_2, \dots\}$

Bounded sequences in $\mathbb{R}$ $V = \{(x_1, x_2, \dots) \mid \exists M \text{ st } |x_i| < M \ \forall i\}$

Sequences bounded by M in $\mathbb{R}$ $V = \{(x_1, x_2, \dots) \mid |x_i| < M \ \forall i\}$

All functions $\{f \mid f: \mathbb{R} \to \mathbb{R}\}$

$\{f : [0,1] \to \mathbb{R} \text{ st } \int f\,dx > 0\}$

$\{f : [0,1] \to \mathbb{R} \text{ st } \int |f| < \infty\}$

Norms: Given a vector space, they measure size of elements $\|X\|$

Applications: Show quality of an approximation.

If $\|X_n - X_*\| \to 0$ as $n \to \infty$, want to conclude $X_n \to X_*$

Need: $\|X\|=0 \Rightarrow X=0$.