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Analysis I
Paul E. Hand
hand@rice.edu

Day 3 — Summary — Limits and continuity of functions

1. Let f be a function defined on $S \subset \mathbb{R}$. The limit of $f(x)$ as x approaches a exists if there exists an L such that for all ε there is a $\delta > 0$ such that $|x - a| < \delta \;\Rightarrow\; |f(x) - L| < \varepsilon$ for $x \in S$. We write such a limit as $\lim_{x\to a} f(x) = L$.

2. Limits commute with addition, multiplication, division, and non-strict inequalities

   (a) If $\lim_{x\to a}(cf)(x) = c \lim_{x\to a} f(x)$ for any real c.

   (b) If $\lim_{x\to a}(f + g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$ if both limits on the right exist.

   (c) If $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$ if both limits on the right exist.

   (d) If $\lim_{x\to a}(f/g)(x) = \lim_{x\to a} f(x)/ \lim_{x\to a} g(x)$ if both limits on the right exist and the limit of g is nonzero.

   (e) If $f(x) \leq g(x)$ for all x sufficiently close to a, then $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$, provided both limits on the right exist.

3. The function $f : S \to \mathbb{R}$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$.

4. The function f is continuous on the set S if f is continuous at every point in S.

5. The composition of two continuous functions is continuous.

6. Intermediate value theorem: Let f be continuous on $[a, b]$. For any y satisfying $f(a) < y < f(b)$ or $f(b) < y < f(a)$, there exists an $x \in (a, b)$ such that $f(x) = y$.

7. The function f is uniformly continuous on the set S if for all ε, there exists a $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$. Notice that the dependence of δ on ϵ does not depend on the position within the set. That is what makes it uniform.

8. A continuous function on a closed interval is uniformly continuous.

Warmup:

Example of sequences x_n, y_n satisfying

$x_n \to 0$

$y_n \to 0$

$\frac{x_n}{y_n} \to a$

Example:

$x_n \to 0$

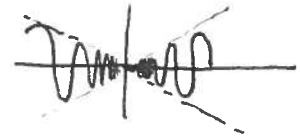
$y_n \to \infty$

$x_n y_n$ has no limit yet is still bounded

1) Let $f: S \to \mathbb{R}$. $\qquad \lim_{x \to a} f(x) = L$ means $\forall \varepsilon \; \exists \delta$ st $|x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Conceptually: f ~~gets~~ is arbitrarily close to L for values of x sufficiently near a.

Example: Let $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ x \sin(\frac{1}{x}) & \text{if } x \neq 0 \end{cases}$



Claim: $\lim_{x \to 0} f(x) = 0$.

Fix ε. Let $\delta = \varepsilon$. If $|x| < \delta$, then

$$|f(x) - L| = \begin{cases} |x \sin\frac{1}{x} - 0| & \text{if } x \neq 0 \\ |0 - 0| & \text{if } x = 0 \end{cases}$$

$$\leq \begin{cases} |x| \, |\sin\frac{1}{x}| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$$\leq |x| < \delta = \varepsilon \; \blacksquare$$

2c) If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$ then $\lim_{x \to a} f(x)g(x) = LM$

finite

Getting at the Proof:

Know: $|f(x) - L|$ small when x near a

$|g(x) - M|$ small when x near a

Want: $|f(x)g(x) - LM|$ small when x near a

Consider $|f(x)g(x) - Lg(x) + Lg(x) - LM|$

$$= |\underbrace{(f(x) - L)}_{\text{small}}\underbrace{g(x)}_{\text{finite}} + \underbrace{L}_{\text{finite}}\underbrace{(g(x) - M)}_{\text{small}}| \leq \frac{\varepsilon}{2C} C + \frac{\varepsilon}{2C} = \varepsilon$$

Need to argue g is bdd.

Let $\varepsilon = 1$. $\exists\, \delta$ st $|x-a| < \delta \Rightarrow |g(x) - M| \leq 1$

$\Rightarrow |g(x)| \leq 1 + |M|$.

Proof: We need to show $\forall \varepsilon\ \exists \delta$ st $|x-a| < \delta \Rightarrow |fg - LM| < \varepsilon$.

By $\lim_{x \to a} g(x) = M$, $\exists\, \delta_0$ st $|g(x)| \leq 1 + M$. for all $|x - x_0| < \delta_0$.

Choose $C = \max(1+M, L)$; Let δ_f & δ_g be from dfn of $\lim_{x\to a} f = L$, $\lim_{x \to a} g = M$ for $\varepsilon/2C$

Fix ε. Let $\delta = \min(\delta_0, \delta_f, \delta_g)$

Then $|f(x)g(x) - LM| = |(f(x) - L)g(x) + L(g(x) - M)|$

$\leq |f(x) - L|\,|g(x)| + |L|\,|g(x) - M|$

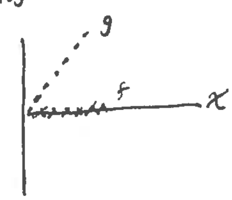
$\leq \frac{\varepsilon}{2C} C + C \frac{\varepsilon}{2C} = \varepsilon$ ∎

2e) $f(x) \le g(x)$ for all $|x-a| < \delta \Rightarrow \lim_{x \to a} f(x) \le \lim_{x \to a} g(x)$ if both limits exist.

Example:

~~Why can't we~~

Can we make a corresponding statement with strict inequalities?



$f(x) < g(x) \Rightarrow \lim f < \lim g$?? No

Let $S = (0, \infty)$

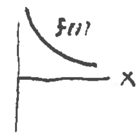
$f(x) = 0, \quad g(x) = x$

On S, $f(x) < g(x)$. But $\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$.

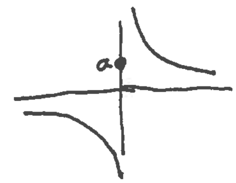
Why dont we say $f(x) \le g(x)$ for all $x \Rightarrow \lim f(x) \le \lim g(x)$ if both limits exist.

With limits as $x \to a$, nothing that is a finite distance away from $x=a$ matters. All that matters is that $f \le g$ "near" $x=a$.

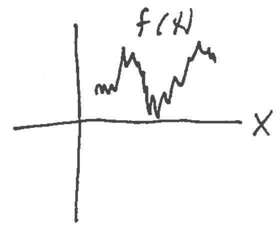
4) Examples: $f(x) = \frac{1}{x}$ is continuous on $(0, \varepsilon)$



$$f(x) = \begin{cases} 1/x & x \neq 0 \\ a & x = 0 \end{cases}$$ is not continuous on $\mathbb{R}$ for any a.



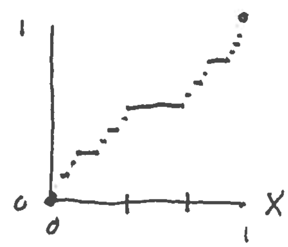
Advanced example: Brownian motion
(continuous random walk)
infinite arc length



Cantor function

continuous & monotonic increasing

Increases despite be flat almost everywhere



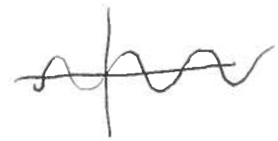
To specify $c(x)$:

1) write x in base 3
2) If x contains a 1, replace all digits subsequent to first 1 by 0
3) Replace all 2's with 1's
4) Interpret as binary #.

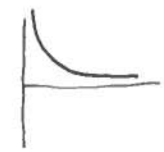
7.8 Uniform Continuity

Examples:

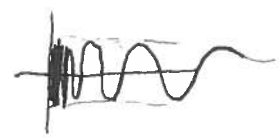
$f(x) = \sin x$ is unif cont on $\mathbb{R}$



$f(x) = \frac{1}{x}$ on $\{x>0\}$ not uniformly continuous



$f(x) = \sin(\frac{1}{x})$ on $\{x>0\}$ not uniformly continuous



Is uniform continuity saying something about slope (if it exists)?

Sort of, but no.

$f(x) = \sqrt{1-x^2}$ is uniformly continuous on $[-1,1]$ and has infinite slope near $x = \pm 1$.