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Analysis I
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Day 26 — Summary — Lebesgue Integral

1. Definition: A simple function is a function that adopts finitely many values: $\phi(x) = \sum_{n=1}^{N} a_n 1_{E_n}$.

2. Define the Lebesgue integral of a nonnegative simple function ϕ as $\int_{\mathbb{R}} f d\mu = \sum_{n=1}^{N} a_n \mu(E_n)$.

3. Define the Lebesgue integral of a nonnegative measurable function f as

$$\int_{\mathbb{R}} f d\mu = \sup \left\{ \int_{\mathbb{R}} \phi d\mu \,\middle|\, \phi \text{ simple, } 0 \leq \phi \leq f \right\}$$

4. Define the Lebesgue integral of a not-necessarily-nonnegative function $f = f^+ - f^-$, where f^+ and f^- are both nonnegative, as

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu,$$

provided both integrals on the right hand side are not infinite.

5. Dominated convergence theorem: Suppose $f_n \to f$ pointwise and $|f_n(x)| < g(x)$ for a nonnegative g such that $\int g d\mu < \infty$. Then $\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n d\mu$

Illustration of Dominated Convergence Thm:

Use Feynman trick to evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$

Let $\displaystyle I(b) = \int_0^{\infty} \frac{\sin x}{x} e^{-bx}\, dx$

By dct
$$I'(b) = -\int_0^{\infty} \sin x \, e^{-bx}\, dx$$
$$= -\frac{1}{1+b^2}$$

So
$$I(t) = -\tan^{-1}(t) + C$$
$$I(\infty) = 0 \Rightarrow C = \pi/2$$
$$I(t) = \pi/2 - \tan^{-1}(t)$$

So $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = 2I(0) = \pi.$

Illustration of DCT:

Show $$\frac{d}{dt}\int_0^\infty \frac{\sin x}{x} e^{-tx}\,dx = -\int_0^\infty \sin x\, e^{-tx}\,dx$$

Proof: $$\lim_{h\to 0} \int_0^\infty \frac{\sin x}{x} e^{-tx}\left(\frac{e^{-hx}-1}{h}\right)$$

Note $|e^{-hx}-1| < |h|\,x$

So integrand dominated by $\sin x\, e^{-tx}$ which has finite integral.

So $$\lim_{h\to 0}\int_0^\infty \frac{\sin x}{x} e^{-tx}\left(\frac{e^{-hx}-1}{h}\right) = \int_0^\infty \sin x\, e^{-tx}(-1)\,dx.$$

Application: Feynman trick for $\int_{-\infty}^\infty \frac{\sin x}{x}\,dx$