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Analysis I
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Day 25 — Summary — Relationship of integrals and derivatives

1. Let f be a regulated map on $[a, b]$. Let $F(x) = \int_a^x f(s)ds$. If f is continuous at the point c, then F is differentiable at c and $F'(c) = f(c)$.

2. Let $f(t, x)$ and $D_2 f(t, x)$ be defined and continuous for $(t, x) \in [a, b] \times [c, d]$. Then, for $x \in [c, d]$, $\frac{d}{dx} \int_a^b f(t, x)dt = \int_a^b D_2 f(t, x)dt$.

Warmup

Plot $\frac{1}{x}\int_0^x e^{-s}\,ds$ for $x>0$ as function of x.

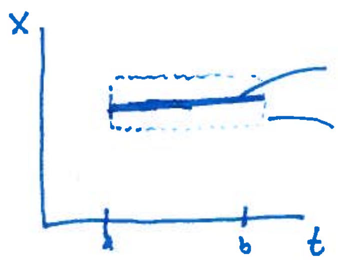
What is its value as $x \to 0$?

2)

Q: When can we interchange derivative & integral?

$$\frac{d}{dx}\int_a^b f(t,x)\,dt = \int_a^b \frac{\partial}{\partial x} f(t,x)\,dt$$

A: when integrand and the relevant partial derivate are continuous in a strip ~~about~~ containing region of integration



region of integration

if f and $\partial_x f$ cont in this strip, then $\frac{d}{dx}\int f\,dt = \int \frac{\partial}{\partial x} f\,dt$

Example: $\frac{d}{dx}\int_1^2 \frac{\sin tx}{t}\,dt = \int_1^2 \cos tx\,dt$

as $\frac{\sin tx}{t}$ & $\cos tx$ are continuous in $[1,2]\times\mathbb{R}$

Formal statement:

Let $f(t,x)$ and $\frac{\partial}{\partial x} f(t,x)$ be in $C^0([a,b]\times[c,d])$

Then $g(x) = \int_a^b f(t,x)\,dt$ is diffable $\forall\, x \in [c,d]$

and $\frac{d}{dx}\int_a^b f(t,x)\,dt = \int_a^b \frac{\partial}{\partial x} f(t,x)\,dt.$

Observe: Thm does not require f to be 2d differentiable (or even 1d diffable in other var)

Let $f(t,x) = |t|$

Note: $f \in C(\mathbb{R}^2)$, $D_2 f \in C(\mathbb{R}^2)$, $D_1 f$ DNE at $t=0$.

By thm $\frac{d}{dx}\int_{-1}^{1} |t|\,dt = \int_{-1}^{1}\frac{d}{dx}|t|\,dt = 0.$

Application: "Feynman trick" for evaluating integrals

Can compute $\int_0^1 \frac{x^2-1}{\log x}\,dx$ by introducing variable & diff'ing under integral

Let $I(b) = \int_0^1 \frac{x^b-1}{\log x}\,dx$.

If $b>-1$, both $\frac{x^b-1}{\log x}$ & $\frac{\partial_b x^b}{\log x}$ are continuous on $[0,1] \times (-1,\infty)$

So $I'(b) = \int_0^1 \frac{\partial}{\partial b} \frac{x^b-1}{\log x}\,dx = \int_0^1 x^b = \frac{1}{b+1}$

So $I(b) = \log(b+1)$. Hence $I(2) = \log 3$.

Note: $\partial_x \frac{x^2-1}{\log x}$ not continuous at $x=0$

Also: can show $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi$ by similar logic. (not Lebesgue integral)

Proof (gist)

Want $\displaystyle\lim_{h\to 0} \int_a^b \frac{f(t,x+h)-f(t,x)}{h}\,dt = \int_a^b \lim_{h\to 0} \frac{f(t,x+h)-f(t,x)}{h}$

We can do this if $\dfrac{f(t,x+h)-f(t,x)}{h} \to \partial_x f(t,x)$ uniformly in t as $h\to 0$

By MVT $\dfrac{f(t,x+h)-f(t,x)}{h} = \partial_x f(t,*)$ for some $*$ near x.

Continuity of $\partial_x f$ on $[a,b]\times[c,d] \Rightarrow$ uniform continuity so if $*$ near x,
then $\partial_x f(t,*)$ near $\partial_x f(t,x)$ (uniformly in t)