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Analysis I
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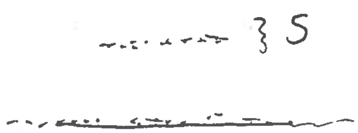
Day 23 — Summary — Lebesgue Measure

1. A measure μ is a mapping from a collection of subsets Σ of $\mathbb{R}$ to the extended reals, satisfying:
   - $E \in \Sigma \Rightarrow \mu(E) \geq 0$
   - $\mu(\emptyset) = 0$
   - Countable additivity: If $\{E_i\}_{i\in\mathbb{N}}$ are pairwise disjoint, then $\mu(\bigcup_{i\in\mathbb{N}} E_i) = \sum_{i\in\mathbb{N}} \mu(E_i)$.
2. If we want $\mu\big((a,b)\big) = b - a$, then there are sets that can not be assigned a measure.
3. Let $E \subset \mathbb{R}$. For an interval $I = (a,b)$, let $|I| = b - a$. The Lebesgue outer measure of E is defined as
$$\mu^*(E) = \inf\left\{ \sum_{k\in\mathbb{N}} |I_k| \;\middle|\; E \subset \bigcup_{k\in\mathbb{N}} I_k, I_k \text{ an open interval} \right\}$$
4. A set E has measure zero if $\forall \varepsilon > 0$, there exist open intervals $\{I_k\}_{k\in\mathbb{N}}$ such that $E \subset \bigcup_{k\in\mathbb{N}} I_k$ and $\sum_{k\in\mathbb{N}} |I_k| \leq \varepsilon$.
5. Any countable set has measure zero.
6. There is an uncountable set of measure zero.
7. Properties of the Lebesgue outer measure:
   - $\mu^*(\emptyset) = 0$.
   - $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$.
   - Countable subadditivity: If $\{E_i\}_{i\in\mathbb{N}}$ is a countable collection of sets, then $\mu^*(\bigcup_{i\in\mathbb{N}} E_i) \leq \sum_{i\in\mathbb{N}} \mu(E_i)$.
   - $\mu^*([a,b]) = b - a$.
8. The Lebesgue outer measure is a measure when restricted to certain subsets of $\mathbb{R}$. These subsets are called measurable sets. The set of measurable sets contains all intervals, is closed under complementation, and is closed under countable unions and countable intersections.

1) What is $\int_{\mathbb{R}} 1_E(x)\,dx$? Should be $1 \cdot$ "length of S"



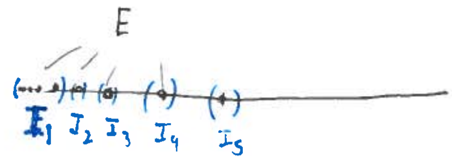
What is size of an arbitrary set? Call it the measure.

$\mu(E)$

Want:

- $\mu[(a,b)] = b-a$
- $\mu(\emptyset) = 0$
- $\mu(E) \geq 0$
- $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$ if $E_1 \cap E_2 = \emptyset$
- $\mu(\bigcup_i E_i) = \sum_i \mu(E_i)$ if E_i are pairwise disjoint

3)



$E \subset \bigcup_{i=1}^{5} I_i$

μ^* is larger than "size" of E : $\mu^*(E) \leq \sum_i |I_i|$

If $\mu^*(E) = a$, then you can ~~take~~ cover the whole set with intervals with combined length no more than $a + \varepsilon$ for any ε.

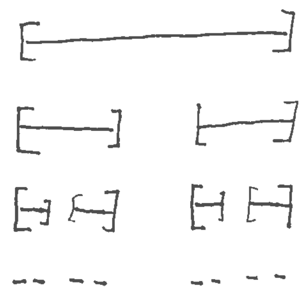
5) Let $E = \{x_i\}_{i=1}^{\infty}$ 

E has measure 0

$$E \subset \bigcup_{i=1}^{\infty} \underbrace{\left(x_i - \frac{\varepsilon}{2^{n+1}}, x_i + \frac{\varepsilon}{2^{n+1}}\right)}_{I_i}$$

$$|I_i| = \frac{1}{2^i} \qquad \sum_{i=1}^{\infty} |I_i| = \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon .$$

6) Middle third cantor set has



E_0 $E_0 = [0,1]$

E_1 $[0,\frac{1}{3}] \cup [\frac{2}{3},1]$

E_2 $[0,\frac{1}{9}] \cup [\frac{2}{9},\frac{1}{3}] + [\frac{6}{9},\frac{7}{9}] \cup [\frac{8}{9},1]$

E_3

$\vdots$

$$E_m = \frac{E_{m-1}}{3} \cup \left(\frac{2}{3} + \frac{E_{m-1}}{3}\right)$$

Let $C = \bigcap_{m=0}^{\infty} E_m$

Middle third cantor set has measure 0

$$C \subset E_m \quad \forall m \qquad |E_m| = \left(\frac{2}{3}\right)^m$$

So $\forall \varepsilon$ $\exists$ E_m finite union of intervals of size $\left(\frac{2}{3}\right)^m < \varepsilon$ for suff large m.

Middle third cantor set is uncountable.

Consists of all reals with only digits 0 or 2 in ternary expansion.
Uncountably many of these