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Analysis I
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Day 19 — Summary — Series within Vector Spaces

1. Let $\sum a_n$ be a series of vectors in a complete normed vector space. If $\sum \|a_n\|$ converges, then $\sum a_n$ converges. The series $\sum a_n$ is said to converge absolutely if $\sum \|a_n\|$ converges.

2. Let $\sum x_n$ be an absolutely convergent series in a complete normed vector space. Then the series obtained by any rearrangement of the series also converges absolutely to the same limit.

3. We say that an infinite series of functions $\sum_n f_n(x)$ converges absolutely on S if $\sum |f_n(x)|$ converges for all $x \in S$. We say the infinite series converges uniformly on S if the sequence of partial sums converges uniformly on S.

4. Weierstrass test: Let $f_n \in L^\infty$ be such that $\|f_n\|_\infty \leq M_n$ and $\sum M_n$ converges. Then $\sum f_n$ converges uniformly and absolutely. If each f_n is continuous, then so is $\sum f_n$.

Warm up

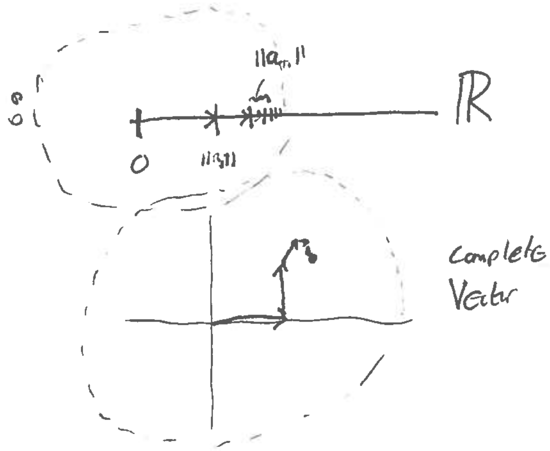
Converge or diverge

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

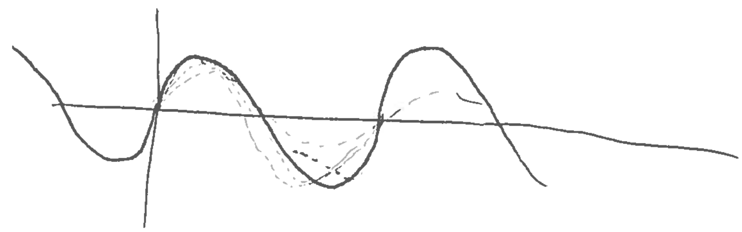
$$\sum_{n=1}^{\infty} \frac{1}{n \log n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n \log^2 n}$$

1) Visually:



Eg $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges in L^{∞}. to a continuous function (with ∞ slope at $x=0$



Thm: If $\sum_{n=1}^{\infty} \|a_n\| < \infty$ then $\sum_{n=1}^{\infty} a_n$ conv (in a complete normed v space)

Pf: Let $S_N = \sum_{k=1}^{n} a_n$ $\quad \|S_n - S_m\| = \|\sum_{k=n+1}^{m} a_n\| \le \sum \|a_n\|$

As $\sum_{n=1}^{\infty} \|a_n\| < \infty$, $\sum_{k=1}^{n} \|a_n\|$ Cauchy, hence $\forall \varepsilon \exists N$ st $n,m \geq N \Rightarrow \|S_n - S_m\| < \varepsilon$.

So S_n Cauchy. complete, so S_n converges.

3) Does $\sum_{n=0}^{\infty} x^n$ converge uniformly on $|x|<1$?

No

Does $\sum_{n=0}^{\infty} x^n$ converge uniform on $|x| < 1-\varepsilon$?

Yes

$\sum_{n=1}^{\infty} x^n$

$\|f^n\| \leq (1-\varepsilon)^n \qquad \sum_{n=0}^{\infty} (1-\varepsilon)^n < \infty.$