16 October 2014
Analysis I
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Day 13 — Summary — Dimensionality of vector spaces

1. Definition: A collection of vectors is linearly dependent if there is a nontrivial linear combination that equals the zero vector.
2. Definition: The span of a collection of vectors is the set of all finite linear combinations of those vectors.
3. Definition: A finite collection of vectors in the space V is a basis if the collection is linearly independent and spans the whole space.
4. If a space has a basis of n elements, than any collection of more than n elements is linearly dependent.
5. If a space has a finite basis, then any collection of vectors that spans V contains a basis.
6. If a space has a basis of n elements, then any collection of n linearly independent elements is a basis.
7. Definition: The dimensionality of a space is the cardinality of any basis. If there is no (finite) basis, then the dimensionality is infinite.

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1.) $X_i \in V \quad i = 1 \dots n$

$\{X_i\}$ lin. independent if $0 = \sum_{i=1}^{n} c_i X_i \Rightarrow c_i = 0 \ \forall i$

$\{X_i\}$ lin. dependent if $\underset{\substack{\text{not all}\\ \text{zero}}}{\exists\, c_i}$ st $0 = \sum_{i=1}^{n} c_i X_i$.

Example: $\begin{pmatrix}1\\1\end{pmatrix} \begin{pmatrix}1\\-1\end{pmatrix} \begin{pmatrix}2\\3\end{pmatrix}$ are linearly dependent

as $\frac{5}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}2\\3\end{pmatrix} = 0$

$\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}$ are linearly independent

as

$$c_1\begin{pmatrix}1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}0\\1\\1\end{pmatrix} + c_3\begin{pmatrix}1\\0\\1\end{pmatrix} = 0$$

$$\Rightarrow \begin{matrix} c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \end{matrix} \Rightarrow c_3 = c_2 \Rightarrow \begin{matrix} c_3 = 0 \\ c_2 = 0 \\ c_1 = 0 \end{matrix}.$$

$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ are linearly dependent as

$$\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix},$$

2) $S \subset V$ (finite or infinite)

$\text{Span}(S) = \{ c_1 v_1 + \dots + c_k v_k \mid k \in \mathbb{Z}^+, c_i \in \mathbb{R}, v_i \in S \}$

set of elements reachable as finite lin. combos of S.

3) If S lin. indep and $\text{span}(S) = V$ then S is <u>basis</u>.

Standard basis of $\mathbb{R}^3$ is $\{\underset{e_1}{(1,0,0)}, \underset{e_2}{(0,1,0)}, \underset{e_3}{(0,0,1)}\}$

— — — $\mathbb{R}^n$ is $\{e_1, e_2, \dots, e_n\}$

Example: A basis for $\text{span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}\right)$ is $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$

5a)

Let V have a basis of n elements.
Any $S \subset V$ that spans V contains a basis.

Proof: Consider $K = \{ k \in \mathbb{Z}^+ \mid$ there is a k-elemeal lin indep subset of $S \}$

If $k \in K$, $k \leq n$. (by (4))

There is a maximal k, call it m.

Let $\{v_1 \cdots v_m\}$ be an lin-indep subset of S. $(*)$

Suppose $v_1 \cdots v_m$ do not span V. $\exists\, w \in S$, $w \notin$ span $v_1 \cdots v_m$.

Then $\{v_1, \cdots, v_m, w\}$ is lin indep.

If not, let $c_0 w + c_1 v_1 + \cdots + c_m v_m = 0$. Either $c_0 = 0$ or $c_0 \neq 0$.

If $c_0 = 0$, then $c_1 \cdots c_m = 0$ by $(*)$

If $c_0 \neq 0$, then $w = \frac{-c_1}{c_0} v_1 \cdots - \frac{c_m}{c_0} v_m$. So $w \in$ span $v_1 \cdots v_m$, contradiction

We thus have a set $\{v_1 \cdots v_m, w\}$ that is lin indep, contradicting m being max of K.

7)

Examples of finite dim space

$\mathbb{R}^n$

$\{$ polynomials of degree $\leq n \}$

Examples of ∞ dim spaces

$\cancel{\ell^\infty}$ $\quad \ell^p \quad 1 \leq p \leq \infty$

$\cancel{L^\infty}$ $\quad L^p \quad 1 \leq p \leq \infty$

$C^0[a,b]$

polynomials