Quiz 1 Practice Solutions

Paul Hand
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1)

$$\left.\begin{array}{l} V_1 = 1 \\ V_5 = 0 \end{array}\right\} \text{given}$$

Node 2: $\dfrac{V_1 - V_2}{R} + \dfrac{V_3 - V_2}{R} = 0$ $\qquad V_1 - 2V_2 + V_3 = 0$

Node 3: $\dfrac{V_2 - V_3}{R} + \dfrac{V_4 - V_3}{R} + \dfrac{V_5 - V_3}{R} = 0$ $\qquad V_2 - 3V_3 + V_4 + V_5 = 0$

Node 4: $\dfrac{V_1 - V_4}{R} + \dfrac{V_3 - V_4}{R} + \dfrac{V_5 - V_4}{R} = 0$ $\qquad V_1 + V_3 - 3V_4 + V_5 = 0$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & a & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2) $A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

subtract $l_{21} = 1$ * first row from second

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix}$$

subtract $l_{32} = 1$ * second row from third

$$\begin{pmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix}$$

subtract $l_{43} = 1$ * 3rd row from 4th

subtract $l_{54} = 1$ * 4th row from 5th

subtract $l_{65} = 1$ * 5th row from 6th

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Putting l's into matrix

$$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}}_{U}$$

3) $z = 1$ — Farth row

$y + z = 2 \Rightarrow y = 1$ — Third row

— Second row

$y + 2z = 3$ ✓

$w + x + y + z = 4 \Rightarrow w + x + 1 + 1 = 4$

$\Rightarrow w + x = 2$

All solutions are given by

$$\begin{pmatrix} w \\ 2-w \\ 1 \\ 1 \end{pmatrix} \text{ for any } w$$

4 a) $v = 0$ Any values of x and z are permissable.

$w = 0$

$0 = 0$

$y = 0$ There are ∞ly many solutions.

$0 = 0$

The matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ has rank 3. (3^{rd} & 5^{th} cols are zero. Other 3 are independent)

By Fund. Thm of Linear algebra,
$N(A)$ has dimensionality 2.

There are two indep. solns.

All solutions are of form $\begin{pmatrix} 0 \\ 0 \\ x \\ 0 \\ z \end{pmatrix}$ for any x, z.

$$= x\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

two linearly independent solutions

b)

$$x+y+z=0$$

Any value of x and y give rise to a z that allows this Eqn to be satisfied. ∞ly many solns.

Viewing the Eqn as one for null space of $(1\ 1\ 1)=A$,

$\operatorname{rank}(A)=1$

$\dim N(A)=2$ by Fund. Thm of Lin Algebra

There are 2 indep solutions.

$x+y+z=0$ is the plane perpendicular to $(1,1,1)$

Solutions given by

$$\begin{pmatrix} x \\ y \\ -x-y \end{pmatrix} \quad \text{for any } x, y$$

$$= x\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

basis for $\{x+y+z=0\}$

5) Apply backsubstitution,

The 5^{th} row gives $15z = 20$, so z can be found.

The 4^{th} row gives $13y$ in terms of a const + multiple of z. Hence could solve for exactly one value of y

The 3^{rd} row gives $10x = c_1 + c_2 y + c_3 z$, can solve for exactly one value of x.

repeat for 2^{nd} row & 1^{st} row.

There is exactly one solution.
(because all diagonals were non zero)

6) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Null space of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$

7) Start with finding a basis for plane:

Write out all points on plane

$$\begin{pmatrix} x \\ y \\ -x-y \end{pmatrix}$$

Express in terms of independent parts

$$x \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

Basis $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Now we orthonormalize:

$$U_1 = \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$U_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \right] \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - 1/\sqrt{2} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 1/2 \\ 0 \\ -1/2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \xrightarrow{\text{normalize}} \begin{pmatrix} -1/2 \\ 1 \\ -1/2 \end{pmatrix} \sqrt{\frac{2}{3}} = \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$$

Orthonormal basis is

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \ \& \ \cancel{\begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}} \begin{pmatrix} -1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$$