PSET 6

1) $U(\Delta x) = U(0) + U'(0)\Delta x + U''(0)\frac{\Delta x^2}{2} + \cdots$

a)

$$\frac{U(\Delta x) - U(0)}{\Delta x} = \underbrace{U'(0) + U''(0)\frac{\Delta x}{2} + \cdots}_{U'(0) \text{ + error that is first order in } \Delta x}$$

b)

$$U(0) = U(0)$$
$$U(\Delta x) = U(0) + U'(0)\Delta x + U''(0)\frac{\Delta x^2}{2} + U'''(0)\frac{\Delta x^3}{6} + \cdots$$
$$U(2\Delta x) = U(0) + U'(0)2\Delta x + U''(0)\frac{4\Delta x^2}{2} + U'''(0)\frac{8\Delta x^3}{6} + \cdots$$

To get a second order approximation of $U'(0)$,

$$\frac{C_0 U(0) + C_1 U(\Delta x) + C_2 U(2\Delta x)}{\Delta x} = 0 \cdot U(0) + 1 \cdot U'(0) + 0 \cdot U''(0)$$

Reading off Δx^0 coefficient:

$$C_0 + C_1 + C_2 = 0$$

Reading off Δx^1 coeff:

$$\frac{C_1 \Delta x + 2C_2\Delta x}{\Delta x} = 1 \implies C_1 + 2C_2 = 1$$

Reading off Δx^2 coeff:

$$\frac{C_1\frac{\Delta x^2}{2} + C_2\frac{4\Delta x^2}{2}}{\Delta x} = 0 \implies C_1 + 4C_2 = 0$$

Solving,
$C_0 = -1.5$
$C_1 = 2$
$C_2 = -0.5$

So $U'(0) \approx -\frac{3}{2}U(0) + 2U(\Delta x) - \frac{1}{2}U(2\Delta x)$

2)

$$-\frac{d^2u}{dx^2} = x$$

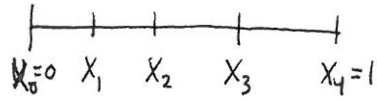
$$u(x) = -\frac{1}{6}x^3 + cx + d$$

$u'(0) = 0 \Rightarrow c = 0$

$u(1) = 0 \Rightarrow d = 1/6$

$$u(x) = \frac{1}{6}(1 - x^3)$$

3)



$N=5 \qquad \Delta X = \frac{1}{4}$ $\qquad$ Let $U_i = U(i\Delta X)$. Let $X_i = i\Delta X$

$$-\frac{d^2U}{dX^2} = X \quad \Longrightarrow \quad \frac{-U_{i+1} + 2U_i - U_{i-1}}{\Delta X^2} = X_i \qquad \text{for } i = 1, 2, 3$$

$$U(1) = 0 \quad \Longrightarrow \quad U_4 = 0$$

$$\frac{dU}{dx}(0) = 0 \quad \Longrightarrow \quad \frac{U_1 - U_0}{\Delta X} = 0$$

So

$$\begin{pmatrix} \frac{-1}{\Delta X} & \frac{1}{\Delta X} & 0 & 0 & 0 \\ \frac{-1}{\Delta X^2} & \frac{2}{\Delta X^2} & \frac{-1}{\Delta X^2} & 0 & 0 \\ 0 & \frac{-1}{\Delta X^2} & \frac{2}{\Delta X^2} & \frac{-1}{\Delta X^2} & 0 \\ 0 & 0 & \frac{-1}{\Delta X^2} & \frac{2}{\Delta X^2} & \frac{-1}{\Delta X^2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 1/2 \\ 3/4 \\ 0 \end{pmatrix}$$

3b) See code

3c)

If $N=100$, max error is $1.7 \cdot 10^{-5}$

If $N=200$ — — — $4.2 \cdot 10^{-6}$

Cutting Δx by 2 cuts error by ≈ 4.

Second order

4a)

$$-\frac{d^2u}{dx^2} = x$$

$$\int_0^1 -\frac{d^2u}{dx^2}\,\phi(x)\,dx = \int_0^1 x\,\phi(x)\,dx$$

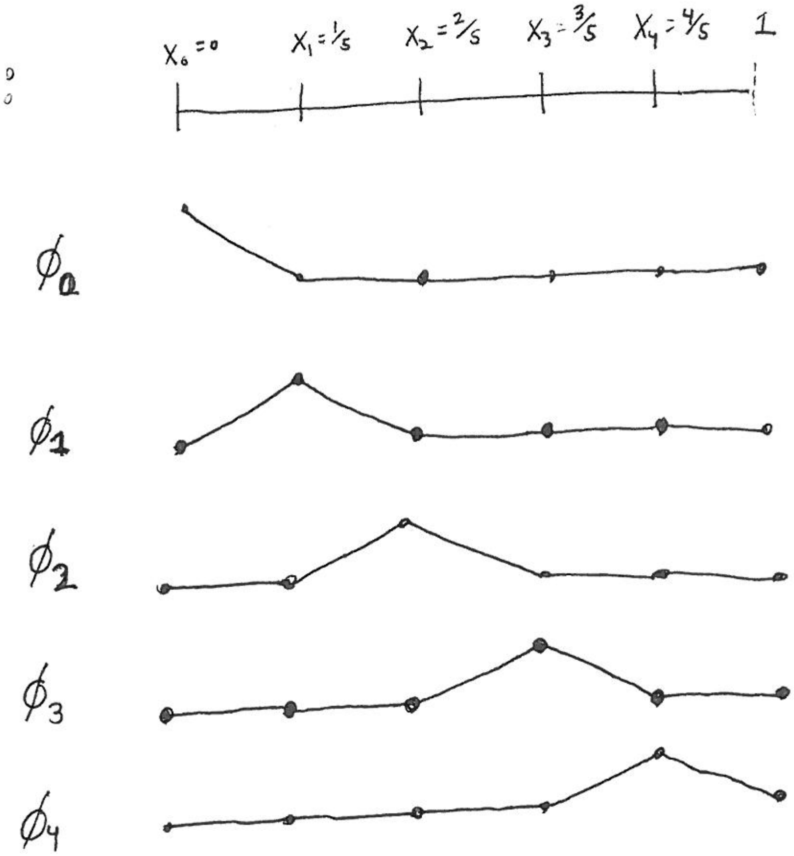
Integrate by parts

$$\int_0^1 \frac{du}{dx}\frac{d\phi}{dx}\,dx - \frac{du}{dx}\phi(x)\Big|_0^1 = \int_0^1 x\,\phi(x)\,dx$$

If $\phi(1)=0$ and $\frac{du}{dx}(0)=0$, boundary term cancels

$$\int_0^1 \frac{du}{dx}\frac{d\phi}{dx}\,dx = \int_0^1 x\,\phi(x)\,dx \quad \text{for all } \phi \text{ such that } \phi(1)=0$$

4b:



$$K_{ij} = \int_0^1 \phi_i' \phi_j' \, dx = \begin{cases} 0 & \text{if } |i-j| > 1 \\ \frac{-1}{\Delta x}\frac{1}{\Delta x}\Delta x = \frac{-1}{\Delta x} & \text{if } |i-j| = 1 \\ \frac{1}{\Delta x}\frac{1}{\Delta x} 2\Delta x = \frac{2}{\Delta x} & \text{if } i = j \neq 0 \end{cases}$$

$$\frac{1}{\Delta x}\frac{1}{\Delta x}\Delta x = \frac{1}{\Delta x} \quad \text{if } i = j = 0$$

$$F_0 = \int_0^1 \phi_0 \, x \, dx = \int_0^{\Delta x} \left(1 - \frac{x}{\Delta x}\right) x \, dx = \frac{\Delta x^2}{6}$$

For $i = 1 \dots 4$ $\quad F_i = \int_0^1 \phi_i \, x \, dx =$ center of mass of bar w/ liner density ϕ_i x mass of bar w/ liner densit ϕ_i

By symmetry about $x = x_i$, c.o.m. of $\phi_i = x_i = i\Delta x$

Mass of bar w/ liner density ϕ_i is $\frac{1}{2} 2\Delta x \cdot 1 = \Delta x$

$$F_i = i\,\Delta x^2$$

$KU = F$

$$\begin{pmatrix} \frac{1}{\Delta X} & \frac{-1}{\Delta X} & 0 & 0 & 0 \\ \frac{-1}{\Delta X} & \frac{2}{\Delta X} & \frac{-1}{\Delta X} & 0 & 0 \\ 0 & \frac{-1}{\Delta X} & \frac{2}{\Delta X} & \frac{-1}{\Delta X} & 0 \\ 0 & 0 & \frac{-1}{\Delta X} & \frac{2}{\Delta X} & \frac{-1}{\Delta X} \\ 0 & 0 & 0 & \frac{-1}{\Delta X} & \frac{2}{\Delta X} \end{pmatrix} U = \begin{pmatrix} \Delta x^2/6 \\ \Delta X^2 \\ 2\Delta X^2 \\ 3\Delta X^2 \\ 4\Delta X^2 \end{pmatrix}$$

c) See code

d) The max error for $N=100$ nodes was $5.5 \cdot 10^{-17}$

———————— $N=200$ ———————— $8.5 \cdot 10^{-15}$

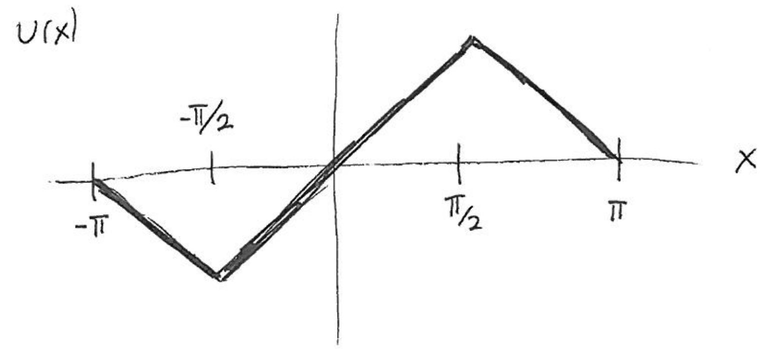
These measurements are at the level of error introduced by rounding, hence we can not compute a meaningful order of accuracy for the method from their ratio.

Because this Finite Element method approximates U by piecewise linear ϕ_i, the largest error away from grid points will by $\sim \Delta x^2$, making the method 2^{nd} order

5) $$-\frac{d^2U}{dx^2} = \delta(x-\pi/2) - \delta(x+\pi/2)$$

$$U(-\pi) = U(\pi)$$
$$U'(-\pi) = U'(\pi)$$

a) like a rubber band where you are pulling up at $x=\pi/2$, down at $x=-\pi/2$



b) $$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx} \qquad \hat{f}(n) = \frac{\int_{-\pi}^{\pi} f(x)\, e^{-inx}}{2\pi}$$

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\delta(x-\pi/2) - \delta(x+\pi/2)\right] e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[e^{-in\pi/2} - e^{in\pi/2}\right] = \frac{-i}{\pi}\;\frac{e^{in\pi/2} - e^{-in\pi/2}}{2i}$$

$$= \frac{-i}{\pi}\sin\frac{n\pi}{2}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{-i}{\pi}\sin\frac{n\pi}{2}\, e^{-inx}$$

c) $$U(x) = \sum_{n=-\infty}^{\infty} \hat{U}(n)\, e^{inx}$$

$$\frac{d^2 U}{dx^2} = \sum_{n=-\infty}^{\infty} \hat{U}(n)\,(-n^2)\, e^{inx}$$

Because
$$-\frac{d^2 U}{dx^2} = f$$
$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx} = \sum_{n=-\infty}^{\infty} n^2 \hat{U}(n)\, e^{inx}$$

$$\hat{f}(n) = n^2 \hat{U}(n) \quad \text{for all } n.$$

So $$\hat{U}(n) = \frac{-i \sin \frac{n\pi}{2}}{\pi n^2} \quad \text{for } n \neq 0$$

$\hat{U}(0)$ is unconstrained.

$\hat{f}(0)$ must be 0 for a soln to exist (which it is)

$$U = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{-i \sin \frac{n\pi}{2}}{\pi n^2}\, e^{inx} + C$$

d) See code