Computational Science and Engineering
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Problem Set 5 Solutions

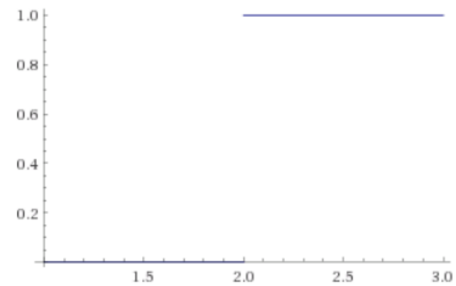
1) (a)

$$\int_{-\infty}^{x} \delta(y-2)dy$$

If x<2, the region of integration does not include singularity, $\int_{-\infty}^{x} \delta(y-2)dy = 0.$

If x>2, the region of integration includes the singularity, $\int_{-\infty}^{x} \delta(y-2)dy = 1.$

So, $\int_{-\infty}^{x} \delta(y-2)dy = H(x-2).$

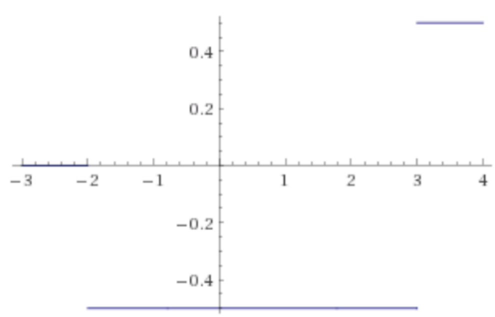


(b)

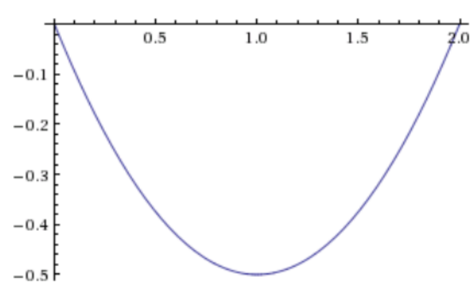
$$\int_{-\infty}^{x} \delta(y-3)dy = H(x-3) \text{ and } \int_{-\infty}^{x} \delta(y+2)dy = H(x+2)$$

So,

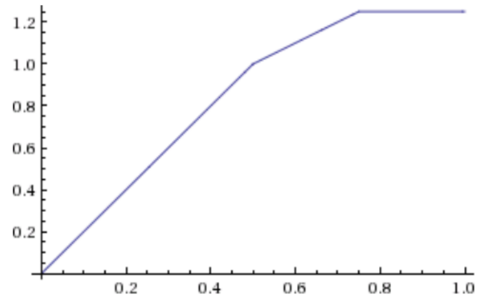
$$\int_{-\infty}^{x} \left(\delta(y-3) - \frac{1}{2}\delta(y+2)\right) dy = H(x-3) - \frac{1}{2}H(x+2).$$



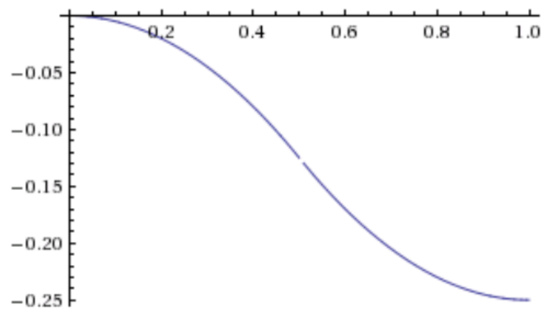
2) (a) Constant force down: f=-1



(b) Slope is 0 on the right, constant except at x=1/2, 3/4. Force is up at x=1/2, 3/4.



(c) Force is up for x<1/2, force is down for x>1/2



To determine that the slope at x=0 equals zero:
Integrate both sides from 0 to 1:

$$\int_0^1 -\frac{d^2u}{dx^2} = \int_0^1 f$$
$$\frac{du}{dx}(0) - \frac{du}{dx}(1) = 0$$

Since $\frac{du}{dx}(1) = 0$, $\frac{du}{dx}(1) = 0$ as well.

3)

$$-\frac{d^2u}{dx^2} = \sin\frac{\pi x}{L}$$
$$u(0) = 0$$
$$u(L) = 0$$

Integrating:

$$u(x) = \frac{L^2}{\pi^2}\sin\frac{\pi x}{L} + cx + d$$

$u(0) = 0$ means $d = 0$

$u(L) = 0 \rightarrow u(L) = \frac{L^2}{\pi^2}\sin\pi + cL = cL = 0$ means $c = 0$.

Solution:

$$u(x) = \frac{L^2}{\pi^2}\sin\frac{\pi x}{L}$$

4)

$$-\frac{d^2u}{dx^2} = \delta(x - L/2)$$
$$u(0) = 0$$
$$u'(L) = 0$$

Integrating:

$$u(x) = ax + b \quad for\ x < \frac{L}{2}$$
$$u(x) = cx + d \quad for\ x > \frac{L}{2}$$

The two sides of the function are linear because $\frac{d^2u}{dx^2} = 0$ away from x=L/2.
$u(0) = 0$ means $b = 0$, $u'(L) = 0$ means $c = 0$.
Now:

$$u(x) = ax\ for\ x < \frac{L}{2}$$
$$u(x) = d\ for\ x > \frac{L}{2}$$

Apply continuity at $x = \frac{L}{2}$:

$$a\frac{L}{2} = d$$

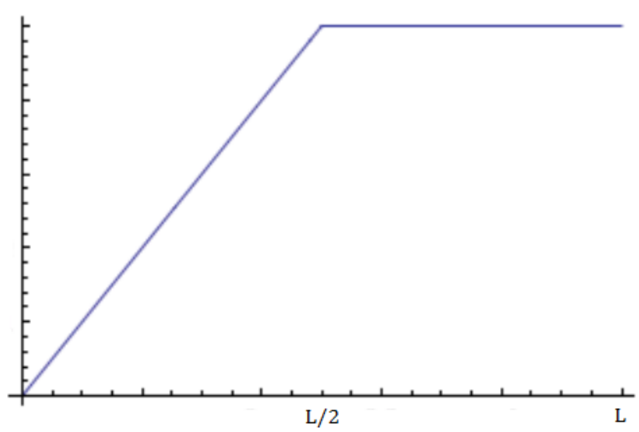
Apply the jump condition:

$$-\left[\frac{du}{dx}\right]_{\frac{L}{2}} = 1$$
$$-(c - a) = 1$$

Since $c = 0$, $a = 1$. Since $aL/2 = d$, $d = L/2$

Solution:

$$u(x) = x \quad for\ x < \frac{L}{2}$$
$$u(x) = \frac{L}{2} \quad for\ x > \frac{L}{2}$$



5) (a) i)

$$\frac{y_{n+1} - y_n}{\Delta t} = Ay_n$$
$$y_{n+1} = y_n + A\Delta t y_n$$

$$\boxed{y_{n+1} = (I + A\Delta t)y_n}$$

ii)

$$\frac{y_{n+1} - y_n}{\Delta t} = Ay_{n+1}$$
$$y_{n+1} - A\Delta t y_{n+1} = y_n$$

$$(I - A\Delta t)y_{n+1} = y_n$$

$$\boxed{y_{n+1} = (I - A\Delta t)\backslash y_n}$$

iii)

$$\frac{y_{n+1} - y_n}{\Delta t} = \frac{Ay_{n+1} + Ay_n}{2}$$

$$y_{n+1} - \frac{\Delta t}{2}Ay_{n+1} = y_n + \frac{\Delta t}{2}Ay_n$$

$$\boxed{y_{n+1} = \left(I - A\frac{\Delta t}{2}\right)\backslash\left(I + A\frac{\Delta t}{2}\right)y_n}$$

b-d)

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A=[-1 0;1 -1];
y0=[1; 0]; %initial y-value, y(0)
dt=0.01; %step size
I=eye(2);
[t, y]=ode45(@(t,y)A*y,[0;1],[1;0]); %arguments: function, timespan, y(0)

y1_forward=(I+A*dt)^(1/(dt))*y0; %where 1/dt is the number of steps to y(1)
y1_trap=((I-A*dt/2)\(I+A*dt/2))^(1/dt)*y0;
y1_ode45=y(end, :)';
```

Results:

y1_forward =	y1_trap =	y1_ode45 =
0.3660	0.3679	0.3679
0.3697	0.3679	0.3679