21 March 2013
18.085
Computational Science and Engineering I
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Problem Set 4

Due: **4 April 2012** in class.

Print or write out any Matlab input and output.

1. (10 points)

   (a) Let x_i be 50 equispaced points from -1 to 1, inclusive. Let $y_i = \frac{1}{3}x_i$. Use Matlab's \ and "vander" commands to try to find the 49th degree polynomial that goes through all (x_i, y_i). What is the fractional error in the computed vector of polynomial coefficients?

   (b) Use Matlab to evaluate the condition number of the matrix from (a). From condition number considerations, how much fractional error should you expect in the computed polynomial coefficients?

2. (10 points) Let

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

   (a) Find the eigenvalue decomposition of A by hand.
   Suggestion: see if you can guess one eigenvector by noticing that both rows add up to 1. Then what must the other eigenvector be?

   (b) Find the eigenvalue decomposition of A^n by multiplying the decomposition of A with itself n times. What matrix does A^n approach as n gets large?

3. (20 points)

   (a) Find a full singular value decomposition of $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \\ 2 & 0 \end{pmatrix}$.
   Hint: What is the rank of A? Range of A? Can you guess an orthonormal basis that maps to an orthonormal basis?

   (b) Let B be given by the following SVD

$$B = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

   Without any calculation, read off an orthonormal basis for the range and an orthonormal basis for the null space.

   (c) Find the eigenvalue decomposition and SVD of the 3×3 matrix C such that Cx is the reflection of x through the plane $x + y + z = 0$.
   Hint: Geometrically identify a basis of eigenvectors

4. (10 points) The jth vector in the Fourier basis is

$$\mathbf{v_j} = \left(e^{2\pi i jk/N}\right)_{k=0,\cdots,N-1}$$

Show that $\mathbf{v_0}, \cdots \mathbf{v_{N-1}}$ are eigenvectors for the $N \times N$ matrix A. Find their corresponding eigenvalues.

$$A = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ -1 & & & -1 & 2 \end{pmatrix}$$

5. (20 points)

   (a) Write out and sketch (by hand or computer) the real and imaginary components of the Fourier basis vectors for $N = 6$.

   (b) By hand, find the fft of

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

   Feel free to verify your answers with Matlab.

6. (10 points) The wave file at `math.mit.edu/~hand/teaching/18.085-spring-2013/single_note_piano.wav` contains a single note played by a piano. Using the Matlab command “wavread”, load the file. It will return a 66150×1 vector corresponding to the waveform sampled at 44100 times per second. Take the fft and determine the frequency (in Hertz) that has maximal amplitude. Use a table of piano key frequencies (e.g. from Wikipedia) to identify which note was played.