Computational Science and Engineering
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February 24, 2013

Problem Set 2 Solutions

1) Show that

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \text{ is the inverse of } S = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix}.$$

Multiply L and S:

$$LS = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} - l_{21} & 1 & 0 \\ l_{31} - l_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $LS = I$, L is the inverse of S.

2)

(a) Find a 2x2 example of $AB \neq BA$.

Let us consider the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = BA.$$

(b) Find a 2x2 example of $A^2 = -I$ with only real entries in A.

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we must find the requirements on a, b, c and d such that $A^2 = -I$ or $A = A^{-1}(-I)$.

$$A^{-1}(-I) = \frac{1}{ad-bc} \begin{bmatrix} -d & b \\ c & -a \end{bmatrix},$$

therefore

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} = A^{-1}(-I).$$

This produces a set of equations:

$$a = \frac{-d}{ad-bc}, \qquad b = \frac{b}{ad-bc}, \qquad c = \frac{c}{ad-bc}, \qquad d = \frac{-a}{ad-bc}.$$

Through some algebra we find the constraints are $a = -d$ and $ad - bc = 1$.

An example: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(c) Find a 2x2 example of $B^2 = 0$ with no zeros in B.

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$B^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + ad & bc + d^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This produces the set of equations:

$$a^2 + bc = 0, \qquad c(a+d) = 0, \qquad b(a+d) = 0, \qquad bc + d^2 = 0$$

Since none of the elements can equal zero, we are left with the constraints $a = -d$ and $bc = -a^2$. An example: $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

3) Start with

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}.$$

Apply elimination to matrix A

$$\underset{\text{(i)}}{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}} \rightarrow \underset{\text{(ii)}}{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 2 & 3 \end{bmatrix}} \rightarrow \underset{\text{(iii)}}{\begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}}.$$

Matrix (ii) is achieved by subtracting $l_{21} = 1/2$ times row 1 from row 2.
Matrix (iii) is achieved by subtracting $l_{32} = 2/3$ times row 2 from row 3.
Hence we have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{bmatrix}, U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 4/3 \end{bmatrix}.$$

4) We first solve for vector **y** such that $Ly = f$, and then solve **x** such that $Ux = y$.
The equation $Ly = f$ is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

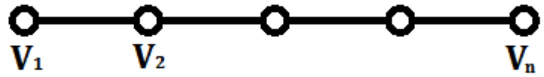
Since L is in lower triangular form, we use back substitution to conclude $y_1 = 0, y_2 = 3, y_3 = 0$.
Now, the equation $Ux = y$ is given by

$$\begin{bmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$$

Since U is in upper triangular form, we use back substitution to conclude $x_1 = -4, x_2 = 1, x_3 = 0$.
Thus,

$$x = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}.$$

5)



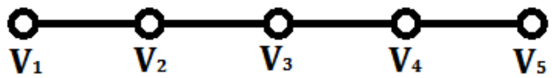
(a) The n equations relating to $v_1, v_2, \ldots v_n$:
Net current into any node is 0:

$$\frac{v_{i-1} - v_i}{R} + \frac{v_{i+1} - v_i}{R} = 0$$

for $i = 2, 3, \ldots n - 1$
Imposed boundary conditions:

$$v_1 = 1$$
$$v_n = 0$$

In the case of n=5



The n equations in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

We can solve this in MATLAB and see that the voltages at the nodes are evenly spaced from 0 to 1:

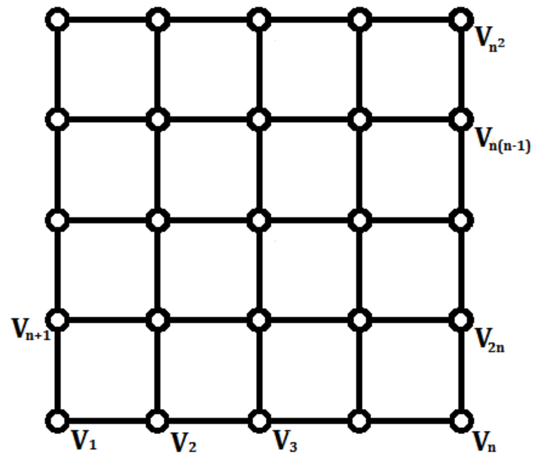
$$v = \begin{bmatrix} 1 \\ 0.75 \\ 0.5 \\ 0.25 \\ 0 \end{bmatrix}$$

(b) MATLAB code:

```
n= 10000;
A=sparse([],[],[],n,n,3*n-4);
b=zeros(n,1);

A(1,1)=1;
A(n,n)=1;
b(1,1)=1;

for i=2:n-1
    A(i,i-1)=1;
    A(i,i)=-2;
    A(i, i+1)=1;
end

%Determine how long it takes to solve Ax=b
tic
x=A\b;
toc

%Print the computed value of node 5000
v5000=sprintf('%0.6f',x(5000));
```

The computed value of v_{5000} is 0.500050. Computation time is 0.002 seconds.

6)



(a) The n^2 equations relating to the nodes:

Net current into any node is 0:

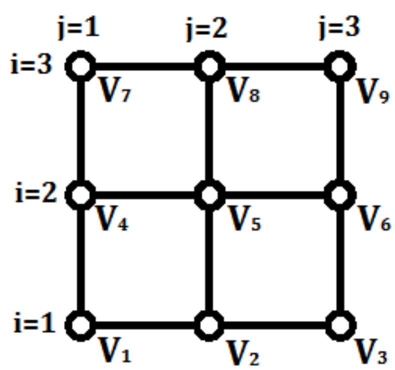
$$\sum_{\substack{j\ a\ neighbor \\ of\ i}} \frac{v_j - v_i}{R} = 0$$

for $i = 2, 3, \ldots n^2 - 1$

Imposed boundary conditions:

$$v_1 = 1$$
$$v_{n^2} = 0$$

In the case of n=3



The n^2 equations in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -3 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \\ v_8 \\ v_9 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

(b) MATLAB code:

```
n=100; %the dimensions of the lattice

A=sparse([],[],[],n^2,n^2,5*n^2);
%Each row of A corresponds to a node in the lattice

ind = @(i,j) (i-1)*n+j;
%the function "ind" translates an (i, j) position in the lattice to a row
%and column index within the matrix A corresponding to the node at (i,j)

%for example:
%In a 3x3 lattice, the location (i, j)=(3, 2) in the lattice corresponds
%to node 8 which is represented by row and column 8 in matrix A.

%Iterate through the nodes:
for i=1:n
    for j=1:n
        c= ind(i,j);
        %We now fill in row 'c' of the matrix corresponding to node 'c'
        if(i==1 && j==1) || (i==n && j==n) %'c' is the first or last node
            A(c, c)=1;
            continue
        end

        if i>1 %node 'c' is not on the bottom row of the lattice
            A(c,c)=A(c,c)-1; %iterate the value of the diagonal by -1
            A(c, ind(i-1, j))= 1; %add 1 in the col of the node below 'c'
        end

        if i<n %node 'c' is not on the top row of the lattice
            A(c,c)=A(c,c)-1; %iterate the value of the diagonal by -1
            A(c, ind(i+1, j))= 1; %add 1 in the col of the node above 'c'
        end

        if j>1 %node 'c' is not in the leftmost column of the lattice
            A(c,c)=A(c,c)-1; %iterate the value of the diagonal by -1
            A(c, ind(i, j-1))= 1; %add 1 in the col of the node left of 'c'
        end

        if j<n %node 'c' is not in the rightmost column of the lattice
            A(c,c)=A(c,c)-1; %iterate the value of the diagonal by -1
            A(c, ind(i, j+1))= 1; %add 1 in the col of the node right of 'c'
        end
    end
end

b=zeros(n^2,1);
b(1)=1;

tic
y=A\b;
toc
v50=sprintf('%0.6f',y(50^2));
```

The computed value of v_{50} is 0.488932.
Computation time is 0.07 seconds.

(c) The 2D problem is much more expensive than the 1D problem.