

## Problem Set 1 Solutions

1) What is the rank of A?

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Because rank is the number of independent rows, and A only has 6 rows, the rank of A is at most 6.

By inspection, 6 independent columns can be identified. For example, columns 1, 3, 4, 5, 7, and 8 are an independent set. We can show that these columns are independent by putting them into a matrix and finding  $\mathbf{c}$ :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving for  $\mathbf{c}$ :

First row says:  $c_1 = 0$                       so  $c_1 = 0$

Fourth row says:  $-c_6 = 0$                       so  $c_6 = 0$

Fifth row says:  $c_5 - c_6 = 0$                       so  $c_5 = 0$

Sixth row says:  $c_4 = 0$                       so  $c_4 = 0$

Third row says:  $c_3 - c_4 = 0$                       so  $c_3 = 0$

Second row says:  $c_2 - c_5 = 0$                       so  $c_2 = 0$

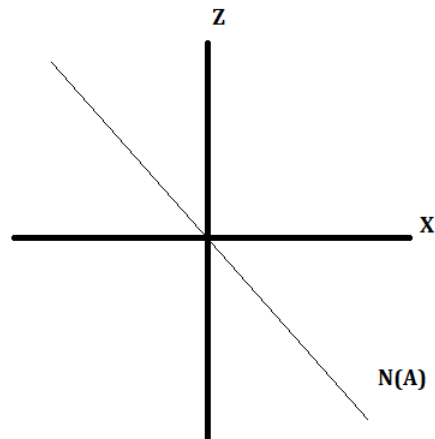
Since  $\mathbf{c} = \mathbf{0}$  is the only solution we know that these columns are independent. This means that the rank of A is at least 6. Since the rank of A is at most 6 and at least 6, we have determined that  $\text{rank}(A) = 6$ .

2) Find and sketch the null spaces of the following matrices:

(a)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We see that  $x + z = 0$ ,  $y = 0$ . The null space is all  $(x, y, z)$  that satisfy these two equations.  $y = 0$  means that the null space is in the  $xz$  plane.  $x + z = 0$  means that  $z = -x$  within that plane.

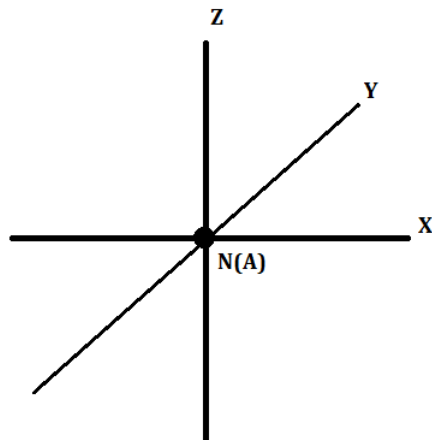


$N(A)$  is a line.  $N(A) = \{ (x, 0, -x) \text{ for all } x \}$ .

(b)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We see that  $x + z = 0$ ,  $y = 0$ ,  $x + y = 0$ . The only point satisfying these equations is  $\mathbf{x} = \mathbf{0}$ . The null space is a point at the origin.



$N(A)$  is a point.  $N(A) = (0, 0, 0)$ .

3) Rewrite the equation as:

$$\begin{bmatrix} 6 \\ 4 \\ 5 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix}$$

Using MATLAB:

```
%define the matrix
```

```
A=[1 0 0 0 1;  
  1 1 0 0 0;  
  0 1 1 0 0;  
  0 0 1 1 0;  
  0 0 0 1 1];
```

```
%define the LHS
```

```
b=[6 4 5 6 9]';
```

```
%x is the vector containing the c values
```

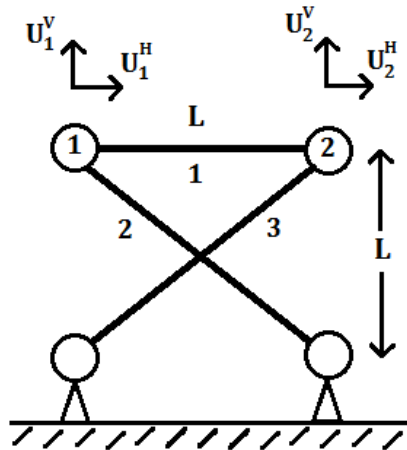
```
x=A\b
```

Result:

```
x =
```

```
 1  
 3  
 2  
 4  
 5
```

4) Consider the following truss:



(a)

$$\text{Bar 1: } -U_1^H + U_2^H = 0$$

$$\text{Bar 2: } -U_1^H \cos \frac{\pi}{4} + U_1^v \sin \frac{\pi}{4} = 0$$

$$\text{Bar 3: } U_2^H \cos \frac{\pi}{4} + U_2^v \sin \frac{\pi}{4} = 0$$

The left hand side is the change in bar length given displacements  $U_1^H$ ,  $U_1^v$ ,  $U_2^H$ , and  $U_2^v$ . The right hand side says that those changes are all zero.

Writing these in matrix form  $A\mathbf{x} = \mathbf{0}$ :

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} U_1^H \\ U_1^v \\ U_2^H \\ U_2^v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

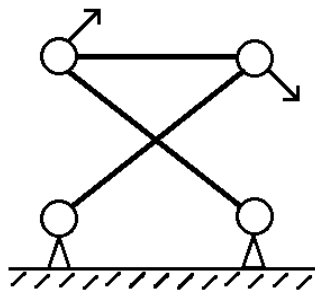
A has rank 3: 1<sup>st</sup>, 2<sup>nd</sup>, and 4<sup>th</sup> columns are independent. By the fundamental theorem of linear algebra:

$$\text{rank}(A) - \dim(N(A)) = 4$$

$$\dim(N(A)) = 1$$

This means there is 1 mode of deformation.

b)



This gives (1, 1, 1, -1) in  $N(A)$ .

Verify:

$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(c) Using MATLAB:

```
a=sqrt(2)/2;
```

```
A=[-1 0 1 0;  
    0 0 a a;  
   -a a 0 0];
```

```
%each row represents a bar and how movements of the nodes affect its length  
%each column is a movement of the nodes
```

```
%the null space of A is the vector of node movements that result in no net  
%change in the length of any of the bars
```

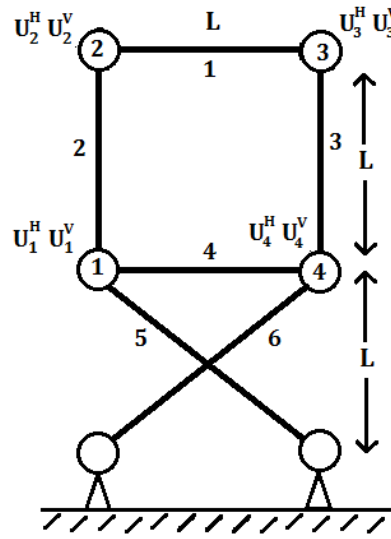
```
x=null(A)
```

Result:

```
x =  
-0.5000  
-0.5000  
-0.5000  
 0.5000
```

This result is -1/2 times the vector that we guessed so the MATLAB answer agrees with the answer from part (b).

5) Consider the following truss:

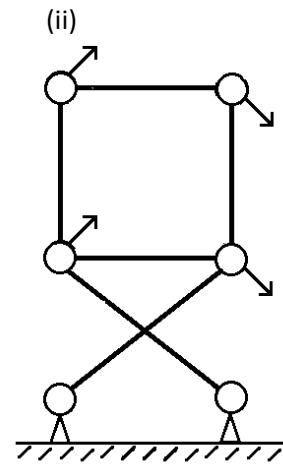
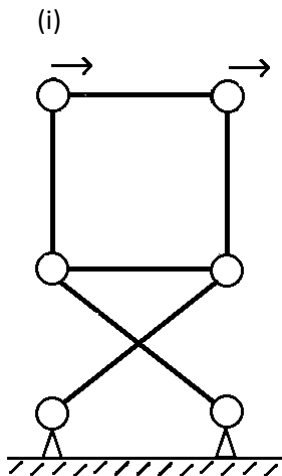


(a) Matrix equation  $Ax = 0$ :

$$\begin{bmatrix} 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} U_1^H \\ U_1^V \\ U_2^H \\ U_2^V \\ U_3^H \\ U_3^V \\ U_4^H \\ U_4^V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A has rank 6. By the fundamental theorem of linear algebra, A has a 2-dimensional null space. This means that there are two independent modes of deformation.

(b)



(i) Translation of bar 1, nodes 1 and 4 remain fixed.  $\mathbf{x} = (0, 0, 1, 0, 1, 0, 0, 0)$

(ii) Rotation of bar 4, square on top maintains shape.  $\mathbf{x} = (1, 1, 1, 1, 1, -1, 1, -1)$

(c) Using MATLAB:

```
a=sqrt(2)/2;
```

```
A=[0 0 -1 0 1 0 0 0;  
  0 -1 0 1 0 0 0 0;  
  0 0 0 0 0 1 0 -1;  
 -1 0 0 0 0 0 1 0;  
 -a a 0 0 0 0 0 0;  
  0 0 0 0 0 0 a a];
```

```
x=null(A)
```

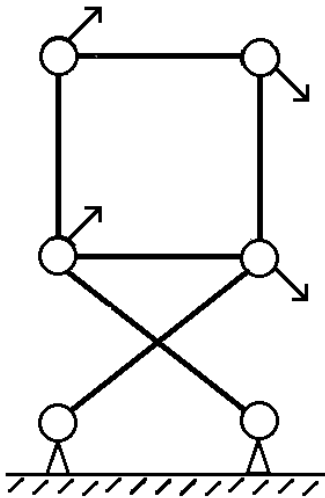
Result:

x =

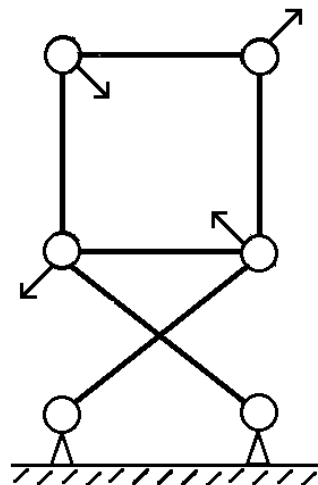
```
0.3381 -0.2288  
0.3381 -0.2288  
0.3963 0.5856  
0.3381 -0.2288  
0.3963 0.5856  
-0.3381 0.2288  
0.3381 -0.2288  
-0.3381 0.2288
```

Visually:

(i)



(ii)



Neither of these are our guesses from part (b). However, we can determine that our guesses are linear combinations of these results.

In other words, we can determine that there is  $\mathbf{c}_1$  such that:

$$\begin{bmatrix} 0.3831 & -0.2288 \\ 0.3831 & -0.2288 \\ 0.3963 & 0.5856 \\ 0.3831 & -0.2288 \\ 0.3963 & 0.5856 \\ -0.3831 & 0.2288 \\ 0.3831 & -0.2288 \\ -0.3831 & 0.2288 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and  $\mathbf{c}_2$  such that:

$$\begin{bmatrix} 0.3831 & -0.2288 \\ 0.3831 & -0.2288 \\ 0.3963 & 0.5856 \\ 0.3831 & -0.2288 \\ 0.3963 & 0.5856 \\ -0.3831 & 0.2288 \\ 0.3831 & -0.2288 \\ -0.3831 & 0.2288 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

Using MATLAB:

```
%solution (i) from part b
b_i=[0,0,1,0,1,0,0,0]';
```

```
c_i=x\b_i
%where x is the null space we calculated in 5(c).
```

```
%solution (ii) from part b
b_ii=[1,1,1,1,1,-1,1,-1]';
```

```
c_ii=x\b_ii
%where x is the null space we calculated in 5(c).
```

Result:

```
c_i =
    0.7926
    1.1712
```

```
c_ii =
    2.8212
   -0.2017
```