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18.085
Computational Science and Engineering I
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Problem Set 4

Due: **5 April 2012** in class.

Print or write out any Matlab input and output.

1. (10 points)

   (a) Let x_i be 50 equispaced points from -1 to 1, inclusive. Let $y_i = x_i$. Use Matlab's $\backslash$ and "vander" commands to try to find the 49th degree polynomial that goes through all (x_i, y_i). What is the fractional error in the vector of polynomial coefficients?

   (b) Same as (a), but with $y_i = \frac{1}{3}x_i$.

   (c) Using condition numbers, explain the results of (b). Use the Matlab command 'cond' to estimate the condition number of a matrix. Explain what makes (a) work when (b) fails.

2. (10 points) Let

$$A = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}.$$

   (a) Find the eigenvalue decomposition of A by hand.
   Suggestion: see if you can guess one eigenvector by noticing that both rows add up to 1. Then what must the other eigenvector be?

   (b) Find the eigenvalue decomposition of A^n by multiplying the decomposition of A with itself n times. What matrix does A^n approach as n gets large?

3. (10 points)

   (a) Find a full singular value decomposition of $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$
   Hint: what is the rank of A? range of A? How does it act upon $(0,1)^t$?

   (b) Let C be given by the following SVD

$$C = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

   Without any calculation, read off an orthonormal basis for the null space of C.

4. (10 points) The jth vector in the Fourier basis is

$$\mathbf{v_j} = \left(e^{2\pi ijk/N}\right)_{k=0,\cdots,N-1}$$

Show that $\mathbf{v_0}, \cdots \mathbf{v_{N-1}}$ are eigenvectors for the $N \times N$ matrix A. Find their corresponding eigenvalues.

$$A = \begin{pmatrix} 2 & -1 & & & -1 \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ -1 & & & -1 & 2 \end{pmatrix}$$

5. (10 points)

   (a) Write out and sketch the real and imaginary components of the Fourier basis for $N = 6$.

   (b) Without a computer, find the fft of

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

6. (10 points) The wave file at `math.mit.edu/~hand/teaching/18.085-spring-2012/single_note_piano.wav` contains a single note played by a piano. Using the Matlab command "wavread", load the file. It will return a 66150×1 vector corresponding to the waveform sampled at 44100 times per second. Take the fft and determine the frequency (in Hertz) that has maximal amplitude. Use a table of piano key frequencies (e.g. from Wikipedia) to identify which note was played.