Ex.1 Let

$$S = \begin{pmatrix} 1 & 0 & 0 \\ f_1 & 1 & 0 \\ f_2 & 0 & 1 \end{pmatrix}, S' = \begin{pmatrix} 1 & 0 & 0 \\ -f_1 & 1 & 0 \\ -f_2 & 0 & 1 \end{pmatrix}.$$

Then

$$SS' = \begin{pmatrix} 1 & 0 & 0 \\ f_1 - f_1 & 1 & 0 \\ f_2 - f_2 & 0 & 1 \end{pmatrix} = Id, S'S = \begin{pmatrix} 1 & 0 & 0 \\ -f_1 + f_1 & 1 & 0 \\ -f_2 + f_2 & 0 & 1 \end{pmatrix} = Id.$$

Ex.2 Let us consider the matrices

$$L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then,

$$LU = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = UL.$$

Let us consider the matrix

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

then one immediately verifies that $A^2 = -Id$.
Let us consider the matrix

$$B = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix},$$

then one immediately verifies that $B^2 = 0$.

Ex. 3 We use the factorization method given in the textbook. First set

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{pmatrix}.$$

Then apply elimination to matrix A.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{\ell_{21}=\frac{1}{2}} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{\ell_{32}=\frac{2}{3}} \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

Hence we have

$$L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}.$$

Ex. 4 We first solve vector y such that $Ly = f$, and then solve x such that $Ux = y$. The equation $Ly = f$ is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}.$$

Since L is of lower triangular form, we use back substitution to conclude $y_1 = 0, y_2 = 3, y_3 = 0$. Now, the equation $Ux = y$ is given by

$$\begin{pmatrix} 2 & 8 & 0 \\ 0 & 3 & 5 \\ 0 & 0 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}.$$

U is of upper triangular form. We use back substitution to conclude $x_3 = 0, x_2 = 1, x_1 = -4$. Thus

$$x = \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}.$$